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STABILITY OF MARKOVIAN PROCESSES III: FOSTER–LYAPUNOV CRITERIA FOR CONTINUOUS-TIME PROCESSES

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Abstract

In Part I we developed stability concepts for discrete chains, together with Foster–Lyapunov criteria for them to hold. Part II was devoted to developing related stability concepts for continuous-time processes. In this paper we develop criteria for these forms of stability for continuous-parameter Markovian processes on general state spaces, based on Foster–Lyapunov inequalities for the extended generator.

Such test function criteria are found for non-explosivity, non-evanescence, Harris recurrence, and positive Harris recurrence. These results are proved by systematic application of Dynkin's formula.

We also strengthen known ergodic theorems, and especially exponential ergodic results, for continuous-time processes. In particular we are able to show that the test function approach provides a criterion for f-norm convergence, and bounding constants for such convergence in the exponential ergodic case.

We apply the criteria to several specific processes, including linear stochastic systems under non-linear feedback, work-modulated queues, general release storage processes and risk processes.

FOSTER'S CRITERION; IRREDUCIBLE MARKOV PROCESSES; STOCHASTIC LYAPUNOV FUNCTIONS; ERGODICITY; EXPONENTIAL ERGODICITY; RECURRENCE; STORAGE MODELS; RISK MODELS; QUEUES; HYPOELLIPTIC DIFFUSION

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1. Introduction

1.1. *Criteria for stability and recurrence.* Our objectives in this paper are to obtain a unified approach to the stability classification of continuous-time Markov processes via Foster–Lyapunov inequalites applied to the generators of the process.

In Part II of this series of papers [25], we developed various such forms of 'stability' for Markov processes. These are analogous to and based on stability concepts in discrete time, developed in Part I [24]. In [24] we also developed (extending [35], [27], [20]) drift or Foster–Lyapunov conditions on the transition

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probability kernels governing the motion of the chain and these served to classify the chain as non-evanescent, recurrent, positive recurrent and so on. Consideration of the generator of the process is natural in continuous time, as the generator is usually more accessible than the transition function.

In this paper we develop a similar approach for stability of general right processes evolving on locally compact separable metric spaces, based upon the extended generator for the process. We obtain criteria for non-explosivity, non-evanescence, Harris recurrence, and positive Harris recurrence, as well as (and perhaps most importantly, in practice) ergodicity and geometric ergodicity. The processes covered by our approach include diffusions and jump-deterministic processes as special cases.

Criteria for stability of continuous-time processes on countable spaces, based on Foster–Lyapunov inequalities (drift conditions) for the infinitesimal generator, have been developed in [34], [36], [6]. For diffusion processes there are precedents for this work to be found in Kushner's work [19], which is primarily concerned with criteria for various generalizations of stability in the sense of Lyapunov using drift conditions associated with the infinitesimal generator, together with the application of Dynkin's formula. Our work is more closely related to that of Khas'minskii [17] who deals with stochastic generalizations of stability in the sense of Lyapunov in a fashion similar to [19], and presents criteria for various forms of recurrence, based again upon Dynkin's formula.

The paper is organized as follows. In the remainder of this section we describe the basic assumptions, present a version of Dynkin's formula and describe a truncation scheme which is required for its application. Section 2 presents a drift condition for the process which is shown to imply 'non-explosion'; that is, that the escape time for the process is infinite with probability 1. In Section 3 a stronger condition is used to obtain non-evanescence of the sample paths of the process. Using results from [25], this gives a criterion for Harris recurrence. In Section 4 we use a continuous-time version of Foster's criterion to obtain sufficient conditions for the existence of an invariant probability π, together with finiteness of $\pi(f)$ for general functions f, and positive Harris recurrence or generalizations of positive recurrence.

Sections 5 and 6 contain the most important results in the paper. We obtain criteria for total variation norm convergence of the distributions of the process, convergence of the expectation of unbounded functions of the process, and criteria under which such convergence takes place at a geometric rate.

In the final part of the paper, we apply all of these results to jump-deterministic processes, including work-modulated queues, general release storage processes and risk processes, and diffusion processes, where we obtain new convergence results for passive linear stochastic systems under static non-linear feedback.

1.2. *The processes $\boldsymbol{\Phi}$ and $\boldsymbol{\Phi}^m$.* Here we provide a brief description of the context which we treat, which is intended to make the paper relatively self-contained. We do not discuss many of the concepts in detail, since the background to this paper and the processes we consider is given in [24] and [25].

We suppose that $\boldsymbol{\Phi} = \{\Phi_t : t \in \mathbb{R}_+\}$ is a time-homogeneous Markov process with state space $(\boldsymbol{X}, \mathscr{B}(\boldsymbol{X}))$, and transition functions (P^t). When $\Phi_0 = x$ the process $\boldsymbol{\Phi}$ evolves on the probability space $(\Omega, \mathscr{F}, \boldsymbol{P}_x)$, where Ω denotes the sample space. It is assumed that the state space $\boldsymbol{X}$ is a locally compact and separable metric space, and that $\mathscr{B}(\boldsymbol{X})$ is the Borel field on $\boldsymbol{X}$.

The operator P^t acts on bounded measurable functions f and σ-finite measures μ on $\boldsymbol{X}$ via

$$P^t f(x) = \int P^t(x, dy) f(y), \qquad \mu P^t(A) = \int \mu(dx) P^t(x, A).$$

For a measurable set A we let

$$\tau_A = \inf\{t \geqq 0 : \Phi_t \in A\}, \qquad \eta_A = \int_0^\infty \boldsymbol{1}\{\Phi_t \in A\}\, dt.$$

The Markov process is called *φ-irreducible* if for the σ-finite measure φ,

$$\varphi\{B\} > 0 \Rightarrow \boldsymbol{E}_x[\eta_B] > 0, \qquad x \in \boldsymbol{X}$$

and Harris recurrent if $\varphi\{B\} > 0 \Rightarrow \boldsymbol{P}_x(\tau_B < \infty) = 1$ for any $x \in \boldsymbol{X}$. Whilst some of our results hold only for irreducible (i.e. φ-irreducible for some φ) processes, many hold without this restriction. The most interesting of our stability results will however be based on Harris recurrence or stronger forms of recurrence.

Throughout this paper we let $\{O_n : n \in \mathbb{Z}_+\}$ denote a fixed family of open precompact sets for which $O_n \uparrow \boldsymbol{X}$ as $n \to \infty$. By *precompact* we mean that the closure of O_n is a compact subset of $\boldsymbol{X}$ for each n. We let $T^m \triangleq \tau_{O_m^c}$ denote the first-entrance time to O_m^c (set to ∞ if the process does not leave the set O_m), and denote by ζ the *exit time for the process*, defined as

$$\zeta \triangleq \lim_{m\to\infty} T^m. \tag{1}$$

We assume without further comment that the process $\{\Phi_t : 0 \leqq t < \zeta\}$ killed at time ζ is a (Borel) *right process* [28].

In the sense of stability used in this paper, the first property of importance is *explosivity*, or rather non-explosivity.

Non-explosivity. We call the process $\boldsymbol{\Phi}$ non-explosive if $\boldsymbol{P}_x\{\zeta = \infty\} = 1$ for all $x \in \boldsymbol{X}$.

The non-explosivity property is often called *regularity* (see [34] in the countable space case, [16] in the piecewise linear context, and [17] for diffusions). Unfortunately, regularity for sets in Markov chain theory can mean something quite different ([26], [23]), so we have adopted this nomenclature, calling the time ζ the time of explosion, essentially as in Kliemann [18].

If the process $\boldsymbol{\Phi}$ is a non-explosive right process, then $\boldsymbol{\Phi}$ is strongly Markovian with right-continuous sample paths, and non-explosivity implies that the set $\{\Phi_t : 0 \leqq t \leqq T\}$ is precompact with probability 1 for any $T \in \mathbb{R}_+$.

In order to develop drift criteria for non-explosivity or recurrence based on the generator of the process introduced in the next section, we need to consider *truncations* of the process $\boldsymbol{\Phi}$.

For $m \in \mathbb{Z}_+$, let Δ_m denote any fixed state in O_m^c and define $\boldsymbol{\Phi}^m$ by

$$\Phi_t^m = \begin{cases} \Phi_t & t < T^m \\ \Delta_m, & t \geqq T^m. \end{cases} \tag{2}$$

Theorem 12.23 of [28] implies that the resulting process is a non-explosive right process. For the theory developed in this paper, we may in fact let $\boldsymbol{\Phi}^m$ denote any non-explosive right process with the property that $\Phi_t^m = \Phi_t$ whenever $t < T^m$. For instance, we may take $\Phi_t^m = \Phi_{t \wedge T^m}$ where $s \wedge t$ denotes the minimum of s and t. This is the approach which is taken in [19]. For applications, however, the specification of a 'graveyard state' Δ_m as in (2) appears most suitable.

1.3. *The extended generator and Dynkin's formula.* Our central goal in this paper is to provide conditions, couched in terms of the defining characteristics of the process $\boldsymbol{\Phi}$, for the various forms of stability developed in [25] to hold.

In general the characteristics used in practice to define the process are not couched in terms of the semigroup P^t, but rather of the *extended generator* of the process. The following definition is a slightly restricted form of that in Davis [9].

The extended generator. We denote by $D(\mathscr{A})$ the set of all functions $V : \mathbf{X} \times \mathbb{R}_+ \to \mathbb{R}$ for which there exists a measurable function $U : \boldsymbol{X} \times \mathbb{R}_+ \to \mathbb{R}$ such that for each $x \in \boldsymbol{X}$, $t > 0$,

$$\boldsymbol{E}_x[V(\Phi_t, t)] = V(x, 0) + \boldsymbol{E}_x\left[\int_0^t U(\Phi_s, s)\, ds\right] \tag{3}$$

$$\int_0^t \boldsymbol{E}_x[|U(\Phi_s, s)|]\, ds < \infty. \tag{4}$$

We write $\mathscr{A}V \triangleq U$ and call $\mathscr{A}$ the extended generator of the process $\boldsymbol{\Phi}$.

The identity (3) states that the adapted process $(M_t^V, \mathscr{F}_t^\Phi)$ is a martingale, where

$$M_t^V = V(\Phi_t, t) - V(\Phi_0, 0) - \int_0^t U(\Phi_s, s) ds.$$

This definition is an extension of the infinitesimal generator (see [19]) for Hunt processes: the more common definition of this is in terms of a differentiation operation as in (5) below.

For general functions f, it is not easy to know if f is in the domain of $\mathscr{A}$. For example, one way to proceed is to first construct the strong generator (see [17] in the context of diffusions), whose domain typically contains bounded functions with appropriate differentiability properties, and then note that the domain of the strong generator is contained in the domain of the extended generator. To circumvent this difficulty and allow the straightforward use of unbounded functions we use a truncation approach which we now describe.

We write $\mathscr{A}_m$ for the extended generator of $\boldsymbol{\Phi}^m$. Under general conditions, $\mathscr{A}_m$ is an extension of $\mathscr{A}$ on the set O_m in the sense that if V is in the domain of $\mathscr{A}$, then it is in the domain of $\mathscr{A}_m$, and on O_m $\mathscr{A}_m V = \mathscr{A} V$. However, such conditions do not concern us, as we do not require that $\mathscr{A}_m$ extend $\mathscr{A}$.

Typically, the domain of $\mathscr{A}_m$ will give us a rich choice of functions for test functions. Three typical examples are:

A. If $\boldsymbol{X}$ is discrete, then the domain of the extended generator for the process (2) includes *any* finite-valued function on $\boldsymbol{X}$. This fact is used in Theorem 7.1 below which gives criteria for exponential ergodicity for a Markov process on a countable state space.

B. If $\boldsymbol{\Phi}$ is an Itô process then the domain of $\mathscr{A}_m$ contains C^2 (the class of functions on $\boldsymbol{X} \times \mathbb{R}_+$ with continuous first and second partial derivatives), while the domain of $\tilde{\mathscr{A}}$ may be far smaller: see Section 8 and [19], [17].

C. Let $\tilde{\mathscr{A}}_m$ denote the *weak infinitesimal generator* for the space-time process $\{(\Phi_t^m, t) : t \in \mathbb{R}_+\}$. A measurable function V on $\boldsymbol{X} \times \mathbb{R}_+$ is in $D(\tilde{\mathscr{A}}_m)$, the domain of $\tilde{\mathscr{A}}_m$, if the limit

$$\tilde{\mathscr{A}}_m V(x, t) \triangleq \lim_{h \downarrow 0} \frac{\boldsymbol{E}_x[V(\Phi_h^m, t+h)] - V(x, t)}{h} \tag{5}$$

exists pointwise and satisfies

$$\lim_{h \downarrow 0} \boldsymbol{E}_x[\tilde{\mathscr{A}}_m V(\Phi_h^m, t+h)] = \tilde{\mathscr{A}}_m V(x, t). \tag{6}$$

If furthermore

$$\sup_{(x,t) \in C} |\tilde{\mathscr{A}}_m V(x, t)| < \infty, \tag{7}$$

whenever $C \subset \boldsymbol{X} \times \mathbb{R}_+$ is compact, then we have that $D(\tilde{\mathscr{A}}_m) \subset D(\mathscr{A}_m)$ (see [19]).

We note that when (7) holds, as it typically will in applications where the generator is derived through the form (5), then the integrability condition (4) is satisfied for the truncated process $\boldsymbol{\Phi}^m$ since the time-integral is almost surely bounded.

Throughout the remainder of this paper we assume that $V : \boldsymbol{X} \to \mathbb{R}_+$ is a positive, measurable function which is in the domain of $\mathscr{A}_m$ for all m. Such a function $V : \boldsymbol{X} \to \mathbb{R}_+$ is called a *norm-like function* if $V(x) \to \infty$ as $x \to \infty$; this means that the level sets $\{x : V(x) \leqq B\}$ are precompact for each $B > 0$. Functions on $\mathbb{R}^k$ which are norm-like include the Euclidean norm $\|\cdot\|$ and any monotone, unbounded function of $\|\cdot\|$.

All of our criteria for stability will rely on a detailed usage of Dynkin's formula, which is a direct consequence of the optional stopping theorem [10].

Dynkin's formula. Let τ be a stopping time for the right process $\boldsymbol{\Phi}$, and suppose that $V: \boldsymbol{X} \times \mathbb{R}_+ \to \mathbb{R}$ is in the domain of the extended generator $\mathscr{A}_m$. Let $\tau^m \triangleq \min\{m, \tau, T^m\}$. Then

$$\boldsymbol{E}_x[V(\Phi^m_{\tau^m}, \tau^m)] = V(x, 0) + \boldsymbol{E}_x\left[\int_0^{\tau^m} \mathscr{A}_m V(\Phi_t, t)\, dt\right], \qquad x \in \boldsymbol{X}. \tag{8}$$

If τ is bounded by a fixed deterministic constant, then we take $\tau^m = \tau \wedge T^m$ in Dynkin's formula without further comment.

A simple but important consequence of Dynkin's formula is the following comparison theorem.

Theorem 1.1 (comparison theorem). Suppose that $\boldsymbol{\Phi}$ is a non-explosive right process, and that V, g_+ and g_- are positive measurable functions. If for each m, $\mathscr{A}_m V \leqq g_+ - g_-$ on O_m then for any $x \in \boldsymbol{X}$, $s \in \mathbb{R}_+$,

$$P^s V(x) + \int_0^s P^t g_-(x)\, dt \leqq V(x) + \int_0^s P^t g_+(x)\, dt.$$

Hence for each $x \in \boldsymbol{X}$,

$$\limsup_{s\to\infty} \frac{1}{s}\int_0^s P^t g_-(x)\, dt \leqq \limsup_{s\to\infty} \frac{1}{s}\int_0^s P^t g_+(x)\, dt$$

$$\liminf_{s\to\infty} \frac{1}{s}\int_0^s P^t g_-(x)\, dt \leqq \liminf_{s\to\infty} \frac{1}{s}\int_0^s P^t g_+(x)\, dt.$$

Proof. For fixed $s \in \mathbb{R}_+$ denote $s^m = s \wedge T^m$. It follows from Dynkin's formula that for $x \in O_m$,

$$\begin{aligned} \boldsymbol{E}_x[V(\Phi^m_{s^m})] &= V(x) + \boldsymbol{E}_x\left[\int_0^{s^m} \mathscr{A}_m V(\Phi_t)\, dt\right] \\ &\leqq V(x) + \boldsymbol{E}_x\left[\int_0^{s^m} \{g_+(\Phi_t) - g_- \wedge m(\Phi_t)\}\, dt\right] \end{aligned} \tag{9}$$

where $g_- \wedge m$ denotes the minimum of g_- and m. We bound g_- in this way to avoid a possibly infinite negative term.

By (9) and the conditions of the theorem, whenever $x \in O_m$,

$$\begin{aligned} \boldsymbol{E}_x[V(\Phi^m_{s^m})] + \boldsymbol{E}_x\left[\int_0^{s^m} g_- \wedge m(\Phi_t)\, dt\right] &\leqq V(x) + \boldsymbol{E}_x\left[\int_0^{s^m} g_+(\Phi_t)\, dt\right] \\ &\leqq V(x) + \boldsymbol{E}_x\left[\int_0^{s} g_+(\Phi_t)\, dt\right]. \end{aligned}$$

Since from non-explosivity $s^m \uparrow s$ as $m \to \infty$, we have from Fatou's lemma

$$P^s V(x) \leqq \liminf_{m\to\infty} \boldsymbol{E}_x[V(\Phi^m_{s^m})].$$

Combining these inequalities, we may apply the monotone convergence theorem to obtain the result.

To apply Dynkin's formula in the comparison theorem we relied on the fact that, under non-explosivity of $\boldsymbol{\Phi}$, we have $\tau^m \to \tau$ as $m \to \infty$, where in this instance $\tau = s$. Since so many of the results of this paper crucially depend on non-explosivity in this way, we first give a general sufficient condition under which non-explosivity holds.

2. Criteria for finite escape times

In this section our aim is to find conditions which ensure that the sample paths of $\boldsymbol{\Phi}$ remain bounded on bounded time intervals, so that the process is non-explosive.

Our first criterion on the extended generator of the process $\boldsymbol{\Phi}$ is the following.

(CD0) *Condition for non-explosion.* There exists a norm-like function V and a constant $c \geqq 0$ such that

$$\mathscr{A}_m V(x) \leqq cV(x) \qquad x \in O_m, \quad m \in \mathbb{Z}_+.$$

It is easy to see that if the apparently weaker bound

$$\mathscr{A}_m V(x) \leqq cV(x) + d \qquad x \in O_m, \quad m \in \mathbb{Z}_+,$$

is satisfied for constants $c,\ d \geqq 0$, then (CD0) is satisfied: if $c > 0$, consider the norm-like function $V + d/c$, which satisfies (CD0).

The abbreviation CD stands for *continuous drift.* The conditions CD we introduce will in general have matching discrete drift conditions DD in [24], but explosion is not a possibility in discrete time so CD0 has no such analogue. We see in the next result that (CD0) puts an upper limit on the rate of positive drift for the process.

Theorem 2.1. If $\boldsymbol{\Phi}$ is a right process and (CD0) is satisfied, then

(i) $\zeta = \infty$, so that $\boldsymbol{\Phi}$ is non-explosive.

(ii) There exists an a.s. finite random variable $\tilde{D}$ such that

$$V(\Phi_t) \leqq \tilde{D} \exp(ct), \qquad 0 \leqq t < \infty. \tag{10}$$

The random variable $\tilde{D}$ satisfies the bound

$$\boldsymbol{P}_x\{\tilde{D} \geqq a\} \leqq \frac{V(x)}{a}, \qquad a > 0, \quad x \in \boldsymbol{X}.$$

(iii) The expectation $\boldsymbol{E}_x[V(\Phi_t)]$ is finite for each x and t, and the following bound holds:

$$\boldsymbol{E}_x[V(\Phi_t)] \leqq \exp(ct) V(x).$$

Proof. We first apply the extended generator $\mathscr{A}_m$ to the function $g(x, t) \triangleq V(x) \exp(-ct)$ to obtain from the product rule that for $x \in O_m$

$$\begin{aligned}\mathscr{A}_m g(x, t) &= \exp(-ct)[\mathscr{A}_m V(x) - cV(x)] \\ &\leqq 0.\end{aligned}$$

The product rule is easily justified, given our integrability condition (4). From Dynkin's formula we have, with $t^m = T^m \wedge t$,

$$\begin{aligned}E_x[g(\Phi^m_{t^m}, t^m)] &= g(x, 0) + E_x\left[\int_0^{t^m} \mathscr{A}_m g(\Phi_s, s)\, ds\right] \\ &\leqq g(x, 0) = V(x).\end{aligned} \tag{11}$$

Let $M_t = \exp(-ct)V(\Phi^m_t)\mathbf{1}(T^m \geqq t)$. We show that the adapted process $(M_t, \mathscr{F}^\Phi_t)$ is a supermartingale. Fix $s < t$, and consider first the event $\{s > T^m\}$. On this event $M_t = M_s = 0$, and hence also

$$E[M_t \mid \mathscr{F}^\Phi_s] = M_s \quad \text{on } \{s > T^m\}.$$

On the event $\{s \leqq T^m\}$ we use (11) to estimate

$$\begin{aligned}E[M_t \mid \mathscr{F}^\Phi_s] &= \exp(-ct)E_{\Phi^m_s}[V(\Phi^m_{t-s})\mathbf{1}\{T^m \geqq t - s\}] \\ &\leqq \exp(-cs)E_{\Phi^m_s}[g(\Phi^m_{(t-s)^m}, (t-s)^m)] \\ &\leqq \exp(-cs)V(\Phi^m_s) = M_s\end{aligned}$$

and so we have the desired supermartingale property. By Kolmogorov's inequality,

$$P_x\left\{\sup_{t \geqq 0} M_t \geqq a\right\} \leqq \frac{V(x)}{a}, \qquad a > 0.$$

Hence, by the definitions,

$$P_x\left\{\sup_{0 \leqq t < T^m} \{V(\Phi_t)\exp(-ct)\} \geqq a\right\} \leqq \frac{V(x)}{a}, \qquad a > 0.$$

Letting $m \to \infty$ and applying the monotone convergence theorem gives

$$P_x\left\{\sup_{0 \leqq t < \zeta} \{V(\Phi_t)\exp(-ct)\} \geqq a\right\} \leqq \frac{V(x)}{a}, \qquad a > 0.$$

Since V is norm-like, we conclude from the definitions that $\zeta = \infty$, which proves (i). The bound above then implies (ii).

To prove (iii) we return to (11). From this bound, Fatou's lemma and non-explosivity we have

$$E_x[\exp(-ct)V(\Phi_t)] \leqq \liminf_{m \to \infty} E_x[g(\Phi^m_{t^m}, t^m)] \leqq V(x),$$

which proves (iii).

A somewhat weaker version of this result is given as Theorem 4.1 of [17] for diffusion processes. When $\boldsymbol{X}$ is countable, then the recent Theorem 1.11 of Chen ([6]; also [5] in Chinese) gives (i), as does Theorem 2.3 of [34] using the much stronger condition that (CD0) holds with $c \leqq 0$. The theorem is also closely related to Lemma 1 of [16] in the jump-linear case, and to Lemma 1 (p. 37) and Theorem 8 (p. 53) of [19], which uses a stronger form of the Lyapunov function V on $\boldsymbol{X}$, again with (CD0) holding with $c \leqq 0$.

3. Criteria for recurrence

3.1. *Criteria for non-evanescence.* In [24] we developed the concept of *non-evanescence*; we say that a trajectory *converges to infinity,* denoted $\{\boldsymbol{\Phi} \to \infty\}$, if $\Phi_t \in C^c$ for any compact set $C \subset \boldsymbol{X}$, and all $t \in \mathbb{R}_+$ sufficiently large, and $\boldsymbol{\Phi}$ is called *non-evanescent* if $\boldsymbol{P}_x\{\boldsymbol{\Phi} \to \infty\} = 0$ for each $x \in \boldsymbol{X}$.

Our next result gives a criterion on the extended generator for the process $\boldsymbol{\Phi}$ to be non-evanescent. This condition will also imply Harris recurrence under suitable continuity conditions on $\boldsymbol{\Phi}$.

(CD1) *Condition for recurrence.* There exists a compact set $C \subset \boldsymbol{X}$, a constant $d > 0$ and a norm-like function V such that

$$\mathscr{A}_m V(x) \leqq d\boldsymbol{1}_C(x), \qquad x \in O_m, \quad m \in \mathbb{Z}_+.$$

Theorem 3.1. (i) If $\boldsymbol{\Phi}$ is a right process and (CD1) holds then $\boldsymbol{\Phi}$ is non-evanescent.

(ii) If $\boldsymbol{\Phi}$ is non-explosive and if

$$\mathscr{A}_m V(x) \leqq d\boldsymbol{1}_C(x), \qquad x \in O_m, \quad m \in \mathbb{Z}_+,$$

then even if V is not a norm-like function we have the bound

$$\boldsymbol{E}_x[V(\Phi_t)] \leqq V(x) + td$$

for any x and t.

Proof. The second result is an immediate consequence of the comparison theorem. To prove (i), first observe that (CD1) implies (CD0), and hence from Theorem 2.1 we see that the process is non-explosive. Suppose now that $\boldsymbol{P}_x\{\boldsymbol{\Phi} \to \infty\} > 0$ for some $x \in \boldsymbol{X}$. Then since C is compact, there exists $r \in \mathbb{R}_+$ such that

$$\boldsymbol{P}_x\{\{\Phi_t \in C^c : t \geqq r\} \cap \{\boldsymbol{\Phi} \to \infty\}\} > 0.$$

Hence defining the probability μ as

$$\mu\{B\} \triangleq \boldsymbol{P}_x\{\Phi_r \in B \mid \Phi_r \in C^c\}, \qquad B \in \mathscr{B}(\boldsymbol{X})$$

we have by the Markov property

$$\boldsymbol{P}_\mu\{\{\Phi_t \in C^c : t \geqq 0\} \cap \{\boldsymbol{\Phi} \to \infty\}\} > 0. \tag{12}$$

We now show that (12) leads to a contradiction. From Dynkin's formula with $\tau = t \wedge \tau_C$, non-explosivity and Fatou's lemma, we get that

$$
\begin{aligned}
\boldsymbol{E}_x[V(\Phi_{t\wedge\tau_C})] &= \boldsymbol{E}_x\left[\liminf_{m\to\infty} V(\Phi^m_{\tau^m})\right] \\
&\leqq \liminf_{m\to\infty} \boldsymbol{E}_x[V(\Phi^m_{\tau^m})] \\
&\leqq V(x) + \liminf_{m\to\infty} \boldsymbol{E}_x\left[\int_0^{\tau^m} \mathscr{A}_m V(\Phi_s)\, ds\right] \\
&\leqq V(x), \qquad x \in \boldsymbol{X}.
\end{aligned}
$$

For all $t > s$ we have $V(\Phi_{t\wedge\tau_C}) = V(\Phi_{s\wedge\tau_C})$ on the event $\{\tau_C \leq s\}$, while for $\tau_C > s$ we may estimate

$$
\begin{aligned}
\boldsymbol{E}[V(\Phi_{t\wedge\tau_C}) \mid \mathscr{F}^{\Phi}_s] &= \boldsymbol{E}_{\Phi_s}[V(\Phi_{(t-s)\wedge\tau_C})] \\
&\leqq V(\Phi_s) = V(\Phi_{s\wedge\tau_C}).
\end{aligned}
$$

Hence $\{V(\Phi_{s\wedge\tau_C}),\ \mathscr{F}^{\Phi}_s : s \geqq 0\}$ is a convergent supermartingale. For any initial distribution, there are two possibilities for the limit V_∞. Either $\tau_C < \infty$, or $\tau_c = \infty$ so that

$$V_\infty = \lim_{t\to\infty} V(\Phi_t) < \infty.$$

Since V is norm-like, this shows that

$$\boldsymbol{P}_\mu\{\{\boldsymbol{\Phi} \to \infty\}^c \cup \{\tau_C < \infty\}\} = 1.$$

This obviously contradicts (12), and completes the proof.

The result is the continuous-time generalization of Theorem 4.5 of [24], and in the case where $\boldsymbol{X}$ is countable, is essentially given (with a far different approach) as Theorem 4 of [36]. It is related to the jump-linear result in Theorem 2 of [16].

3.2. *Criteria for Harris recurrence.* We showed in [24] and [25] that, under appropriate continuity conditions, non-evanescence is equivalent for both discrete and continuous time processes to the seemingly far stronger property of *Harris recurrence,* defined by either of

(i) there exists some σ-finite measure μ, such that whenever $\mu\{A\} > 0$

$$\boldsymbol{P}_x\{\tau_A < \infty\} \equiv 1;$$

(ii) there exists some σ-finite measure φ, such that whenever $\varphi\{A\} > 0$

$$\boldsymbol{P}_x\{\eta_A = \infty\} \equiv 1.$$

These are shown to be equivalent in Theorem 1.1 of [22]. Note that Harris-recurrent chains are irreducible, whereas non-evanescent chains need not be. To link the two concepts, recall from [25] the definition of an embedded discrete-time chain sampled according to a probability distribution a on $\mathbb{R}_+$: this is governed by the Markov transition function K_a defined by

$$K_a \triangleq \int P^t a(dt). \tag{13}$$

Using the idea of sampled chains we define *petite sets*: a non-empty set $C \in \mathscr{B}(X)$ is called *φ_a-petite* if φ_a is a non-trivial measure on $B(X)$ and a is a probability distribution on $(0, \infty)$ which satisfy the bound $K_a(x, \cdot) \geqq \varphi_a(\cdot)$ for all $x \in C$.

The degree of stability of $\boldsymbol{\Phi}$ is shown in [25] and in [22] to depend critically on the distribution of τ_C for petite sets C. Here, we first note the following result.

Theorem 3.2. If all compact subsets of X are petite and (CD1) holds then $\boldsymbol{\Phi}$ is Harris recurrent.

Proof. This follows from Theorem 3.1, in conjunction with Theorem 3.2(a) and Theorem 5.1(i) of [25].

We can achieve the same result using a seemingly different property of sampled chains. Recall from Section 3 of [25] that a right process is called a *T-process* if for some distribution a, the kernel K_a satisfies $K_a(x, A) \geqq T(x, A)$, where the function $T(\cdot, A)$ is lower semi-continuous for each $A \in \mathscr{B}(X)$, and where $T(x, X)$ is non-zero for all $x \in X$.

The condition that all compact subsets of X are petite is shown in Theorem 5.1(i) of [25] to be, for non-evanescent processes, equivalent to the assumption that $\boldsymbol{\Phi}$ is an irreducible T-process. This immediately gives the following result.

Theorem 3.3. If (CD1) holds for an irreducible T-process then $\boldsymbol{\Phi}$ is Harris recurrent.

Several examples of processes shown in [25] to satisfy the T-process conditions, including diffusion processes, storage models and risk models, will be analyzed in later sections.

4. Criteria for positivity and $\pi(f) < \infty$

4.1. *A positive recurrence criterion.* In the previous section we worked from the topological condition of non-evanescence to the probabilistic condition of Harris recurrence. In developing still stronger stability results, it is more rewarding to start from the Harris recurrence viewpoint. It is well known (cf. [1], [12]) that if $\boldsymbol{\Phi}$ is Harris recurrent than an essentially unique invariant measure π exists. If the invariant measure is *finite*, then it may be normalized to be a probability meaure; in this case $\boldsymbol{\Phi}$ is called *positive Harris recurrent.*

The following result follows from Theorems 1.1 and 1.2 of [22]: it shows that bounds on the hitting times of petite sets will be crucial in characterizing positive recurrence, and we will then seek ways to develop these bounds through Foster–Lyapunov criteria on the generator.

Theorem 4.1. Let $C \in \mathscr{B}(\boldsymbol{X})$ be a closed petite set, suppose that $\boldsymbol{P}_x\{\tau_C < \infty\} \equiv 1$, and that for some $\delta > 0$

$$\sup_{x \in C} \boldsymbol{E}_x\left[\int_0^{\tau_C(\delta)} f(\Phi_t)\, dt\right] < \infty, \tag{14}$$

where $\tau_C(\delta) = \delta + \theta^\delta \tau_C$, and $f \geqq 1$. Then $\boldsymbol{\Phi}$ is positive Harris recurrent and $\pi(f) < \infty$.

The following Foster–Lyapunov drift condition, which is stronger than (CD1) in the special case where V is norm-like and C is a compact set, will be shown to yield a criterion for positive Harris recurrence regardless of the structure of V.

(CD2) *Positive recurrence condition.* For some $c, d > 0$, $f \geqq 1$, a measurable set C, and $V \geqq 0$,

$$\mathscr{A}_m V(x) \leqq -cf(x) + d\mathbb{1}_C(x), \qquad x \in O_m, \quad m \in \mathbb{Z}_+.$$

Note that for positivity we do not require V to be norm-like, although without this assumption we will need to verify non-explosivity separately.

As in [24] we may also consider time-varying test functions. We do not give these results in detail here in order to make the results more transparent; but for this one case we note that for a time-varying function $V : \boldsymbol{X} \times \mathbb{R}_+ \to \boldsymbol{X}$ the criterion for positive recurrence becomes the following.

(CD2′) For some $c, d > 0$, $f \geqq 1$, and a measurable set C,

$$\mathscr{A}_m V(\Phi_t, t) \leqq -cf(\Phi_t) + d\mathbb{1}_C(\Phi_t),$$

when $\tau_{O_m^c} > t > 0$, $m \in \mathbb{Z}_+$.

Extensions of all our results to the time-varying case are straightforward given the results here and in [24].

Theorem 4.2. Suppose that $\boldsymbol{\Phi}$ is a non-explosive right process, that (CD2) holds for $\boldsymbol{\Phi}$ with C a closed petite set, and that $V(x)$ is bounded on C. Then the process is positive Harris recurrent, and moreover $\pi(f)$ is finite.

This result extends Theorem 5.4 of [24] to the continuous-time case. There are some precedents for using (CD2) in continuous time: our result extends [36], [6] to general state spaces, although the method of proof is completely different, whilst the 'asymptotic stability' results of Theorems 2 and 6 of [19] exploit a form of (CD2), and Condition (CD2) is also used by Khas'minskii in [17] in his treatment of diffusion processes.

The proof of Theorem 4.2 first requires us to establish a range of consequences of (CD2) used in conjunction with Dynkin's formula and the Comparison Theorem 1.1.

Theorem 4.3. Suppose that $\boldsymbol{\Phi}$ is a non-explosive right process and that (CD2) holds.

(i) For any $x \in \boldsymbol{X}$, $\delta \geqq 0$,

$$\boldsymbol{E}_x\left[\int_0^{\tau_C(\delta)} f(\Phi_t)\,dt\right] \leqq c^{-1}(V(x) + d\delta). \tag{15}$$

(ii) For every x

$$\limsup_{s\to\infty} \frac{1}{s}\int_0^s P^t f(x)\,dt \leqq d/c. \tag{16}$$

If an invariant probability exists, then $\pi(f) \leqq d/c$.
(iii) For every x

$$\liminf_{s\to\infty} \frac{1}{s}\int_0^s P^t(x, C)\,dt \geqq c/d. \tag{17}$$

If an invariant probability exists, then $\pi(C) \geqq c/d$.

Proof. (i) It follows from (CD2) and Dynkin's formula that

$$0 \leqq \boldsymbol{E}_x[V(\Phi^m_{\tau^m_C})] \leqq V(x) - \boldsymbol{E}_x\left[\int_0^{\tau^m_C} cf(\Phi_t)\,dt\right], \qquad x \in O_m \cap C^c;$$

hence by the monotone convergence theorem, non-explosivity, and the fact that $\boldsymbol{P}_x(\tau_C = 0) = 1$ for $x \in C$,

$$\boldsymbol{E}_x\left[\int_0^{\tau_C} cf(\Phi_t)\,dt\right] \leqq V(x), \qquad x \in \boldsymbol{X}$$

so that (15) is proved in the special case where $\delta = 0$.
We have by the Markov property and this inequality

$$\boldsymbol{E}_x\left[\int_\delta^{\tau_C(\delta)} cf(\Phi_t)\,dt\right] = \int P^\delta(x, dy)\boldsymbol{E}_y\left[\int_0^{\tau_C} cf(\Phi_t)\,dt\right] \leqq P^\delta V(x),$$

and from the comparison theorem with $g_- = cf$ and $g_+ = d\boldsymbol{1}_C$ we have

$$P^\delta V(x) + \boldsymbol{E}_x\left[\int_0^\delta cf(\Phi_t)\,dt\right] \leqq V(x) + \boldsymbol{E}_x\left[\int_0^\delta d\boldsymbol{1}\{\Phi_t \in C\}\,dt\right]. \tag{18}$$

Combining these bounds, we have

$$\boldsymbol{E}_x\left[\int_0^{\tau_C(\delta)} cf(\Phi_t)\,dt\right] \leqq V(x) + d\boldsymbol{E}_x\left[\int_0^\delta \boldsymbol{1}\{\Phi_t \in C\}\,dt\right]$$

which gives the bound (15) for arbitrary δ.

(ii) The bound (16) follows from the comparison theorem on letting $g_- = cf$ and $g_+ = d\boldsymbol{1}_C$. If π is any invariant probability, then by stationarity and Fatou's lemma we have for each fixed $0 < L < \infty$,

$$\begin{aligned}\pi(f \wedge L) &= \limsup_{s\to\infty} \int \left\{\frac{1}{s}\int_0^s P^t(f \wedge L)(x)\,dt\right\}\pi(dx)\\ &\leqq \int \left\{\limsup_{s\to\infty} \frac{1}{s}\int_0^s P^t(f \wedge L)(x)\,dt\right\}\pi(dx)\\ &\leqq (d/c)\pi(\boldsymbol{X})\end{aligned}$$

from (16). Letting $L \uparrow \infty$ and applying the monotone convergence theorem completes the proof.

(iii) If (CD2) holds then the conditions of the comparison theorem are satisfied with $g_- \equiv c$ and $g_+ = d\mathbb{1}_C$. We then have for each x,

$$c \leqq \liminf_{s\to\infty} \frac{d}{s}\int_0^s P^t(x, C)\,dt$$

which proves (17). The bound on $\pi(C)$ is then proved using Fatou's lemma as in (ii).

Result (i) was essentially given as Theorem 7.3 of [17] in the case of diffusions on Euclidean space.

Applying Theorem 4.3 and the characterization in Theorem 4.1 we can give the following proof.

Proof of Theorem 4.2. From Theorem 4.3(i) we see that $\tau_C < \infty$ a.s. $[\boldsymbol{P}_*]$, and that the bound (14) holds for any δ. Hence Theorem 4.1 gives the result.

Putting this result together with Theorem 5.1 of [25] gives immediately an important special case for T-processes.

Theorem 4.4. If (CD2) holds for an irreducible non-explosive T-process with C a compact set and with V bounded on C, then $\boldsymbol{\Phi}$ is positive Harris recurrent and $\pi(f) < \infty$.

4.2. *Existence of invariant measures and boundedness in probability.* Even without irreducibility, the methods above may be used to prove the existence of an invariant probability for a Feller Markov process or a T-process. Our first result extends Theorem 2 of [37] from discrete-time to continuous-time processes.

Theorem 4.5. Suppose that $\boldsymbol{\Phi}$ is a non-explosive right process with the Feller property: that is, $P^t g$ is a bounded continuous function whenever g is bounded and continuous. If (CD2) holds for some compact set $C \subset \boldsymbol{X}$, then an invariant probability exists, and $\pi(f) \leqq d/c$ for any invariant probability π.

Proof. A result of Foguel [11], generalized by Stettner in [30], states that for a Feller process, there are two mutually exclusive possibilities: either an invariant probability exists, or

$$\limsup_{r\to\infty} \sup_{\mu} \frac{1}{r}\int_0^r \mu P^s(C)\,ds = 0 \tag{19}$$

for any compact set $C \subset \boldsymbol{X}$, where the supremum is taken over all initial distributions μ on the state space $\boldsymbol{X}$.

The existence of an invariant probability follows directly from Theorem 4.3(iii) and (19), and the bound on $\pi(f)$ follows from Theorem 4.3(ii).

We now consider T-processes in the reducible case.

Theorem 4.6. Suppose that $\boldsymbol{\Phi}$ is a non-explosive T-process. If (CD2) holds for some compact set $C \subset \boldsymbol{X}$, then for some $n \geqq 1$

$$\boldsymbol{X} = \sum_{i=1}^{n} H_i + E$$

where each H_i is a positive Harris set and $\boldsymbol{P}_x\{\eta_E = \infty\} = 0$ for all x. For any invariant probability π we have $\pi(f) < c/d$.

Proof. By Theorem 4.3(i) $\boldsymbol{\Phi}$ is non-evanescent, and hence the Harris part of the Doeblin decomposition in Section 4 of [25] is non-trival. Each Harris set is positive by part (iv) of the Doeblin decomposition theorem 4.1 of [25], and Theorem 4.3(iii). In view of (ii) of that decomposition theorem, and the observation that $\pi\{C\} \geqq c/d$ for any invariant probability by Theorem 4.3(iii), the number of Harris sets must be finite.

The proof that $\pi(f)$ is finite follows from Theorem 4.3(ii).

For T-processes, (CD2) also provides a criterion for a topological stability condition which applies, as does non-evanescence, even in the reducible case.

In [25], the process $\boldsymbol{\Phi}$ was defined to be *bounded in probability on average* if for each initial condition $x \in \boldsymbol{X}$ and each $\varepsilon > 0$, there exists a compact subset $C \subset \boldsymbol{X}$ such that

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \boldsymbol{P}_x\{\Phi_s \in C\}\,ds \geqq 1 - \varepsilon.$$

Conditions implying boundedness in probability on average are given in Theorem 3.2(b) of [25]. These immediately give the following result.

Theorem 4.7. Suppose $\boldsymbol{\Phi}$ is a non-explosive T-process. If (CD2) holds for some compact set $C \subset \boldsymbol{X}$ then $\boldsymbol{\Phi}$ is bounded in probability on average.

4.3. *Explosive processes under* (CD2). One might hope that the stronger drift condition (CD2) implies the process is non-explosive even when the test function V is not norm-like.

This is not true. When $\boldsymbol{X}$ is countable, then Theorem 2.3 of [34] provides an example of an irreducible birth and death process satisfying (CD2) with V not norm-like, for which non-explosivity fails. On a continuous space, we can construct the following even simpler counterexample.

Let $\boldsymbol{\Phi}$ be a jump-deterministic process on $[0, \infty)$ which follows the deterministic trajectory $w(t)$ between the time points $\{t_i\}$ of a Poisson process of rate λ. Suppose that w is a continuous, strictly increasing function on $[0, 1)$ with $w(0) = 0$ and $w(t) \to \infty$ as $t \to 1$. Let $\boldsymbol{\Phi}$ jump to $\{0\}$ at t_i, $\Phi_{t_i+s} = w(s)$ for $t_i \leqq s < t_{i+1}$, and the process is killed at $\zeta = \lim_{m\to\infty} T^m$, as usual. Clearly, $\boldsymbol{P}_0(\zeta = 1) = \exp(-\lambda) > 0$; and indeed, by a geometric trials argument $\boldsymbol{P}_x(\zeta < \infty) = 1$ for all x. Now let $V : \mathbb{R}_+ \to [0, 1]$ denote any smooth increasing function for which $V(x) = 1$ for $x \geqq 1$ and $V(0) = 0$. For any m, we have $\mathscr{A}_m V(x) = -\lambda$ for $x \geqq 1$, and so, although $\boldsymbol{\Phi}$ is explosive, (CD2) holds for this bounded V.

In most applications the test function V in (CD2) will be norm-like, in which case (CD0) is satisfied and then Theorem 2.1 immediately implies that $\boldsymbol{\Phi}$ is non-explosive. We conjecture that a process of 'rapid' explosion, with $\boldsymbol{E}_x(\zeta)$ getting vanishingly small for x near infinity as in this example, is the only way for explosion to occur under (CD2).

5. Criteria for convergence in total variation

The Markov process $\boldsymbol{\Phi}$ is called *ergodic* if an invariant probability π exists and

$$\lim_{t\to\infty} \|P^t(x, \cdot) - \pi\| = 0, \qquad x \in \boldsymbol{X}.$$

In [25] we saw that if any one skeleton chain is irreducible, then positive Harris recurrence and ergodicity are equivalent concepts. This result leads us to define the following condition on the chain $\boldsymbol{\Phi}$:

($\mathscr{S}$) The chain $\boldsymbol{\Phi}$ is a non-explosive right process, all compact sets are petite for some skeleton chain, and (CD2) holds for some compact set C with V bounded on C.

We immediately have the following result.

Theorem 5.1. Suppose that ($\mathscr{S}$) holds. Then $\boldsymbol{\Phi}$ is ergodic.

Proof. From Theorem 4.2 we see that $\boldsymbol{\Phi}$ is positive Harris recurrent, and from Theorem 5.1(i) of [25] we know that the skeleton chain is irreducible when all compact sets are petite for the skeleton. These two properties are shown to imply ergodicity in Theorem 5.1 of [25].

Observe that although the conditions of this theorem imply that $\pi(f)$ is finite, where f is the function used in (CD2), we cannot in general say if $\boldsymbol{E}_x[f(\Phi_t)]$ converges to $\pi(f)$ as $t \to \infty$; we do not even know if this expectation is bounded in t. This is in contrast to the discrete-time situation.

We now search for criteria for convergence of the expectation $\boldsymbol{E}[f(\Phi_t)]$ to a steady state value. To approach this question we need the concept of the f-norm $\|\mu\|_f$. For any positive measurable function $f \geqq 1$ and any signed measure μ on $\mathscr{B}(\boldsymbol{X})$ we write

$$\|\mu\|_f = \sup_{|g| \leqq f} |\mu(g)|.$$

In [24] we showed that the discrete-time analogue of (CD2) implies f-norm convergence of the distributions for each initial condition: that is,

$$\lim_{n \to \infty} \|P^n(x, \cdot) - \pi\|_f = 0, \qquad x \in \boldsymbol{X}.$$

In particular this shows that this expectation does converge for discrete-time processes. In continuous time we can at least show that under ($\mathscr{S}$) there are functions related to f whose expectations do converge without further conditions. Let $f \geqq 1$ and define for any Δ the function f_Δ by

$$f_\Delta(x) = \int_0^\Delta P^s f(x)\, ds.$$

Theorem 5.2. Suppose that ($\mathscr{S}$) holds. If π denotes the unique invariant probability measure for $\boldsymbol{\Phi}$ we have

$$\lim_{t \to \infty} \|P^t(x, \cdot) - \pi\|_{f_\Delta} = 0, \qquad \forall x \in \boldsymbol{X}.$$

Proof. The chain is ergodic and so π exists from Theorem 5.1. From Theorem 4.3 the hypotheses of Proposition 6.2 of [25] are satisfied, and the result holds.

There are a number of ways in which we can ensure that the desired limit holds in the f-norm itself under the conditions of Theorem 5.2.

Theorem 5.3. Suppose that ($\mathscr{S}$) holds and that one of the following conditions on f holds:

(i) for some constant $\delta > 0$, and constants c_δ, $d_\delta < \infty$,

$$P^s f \leqq c_\delta f + d_\delta, \qquad 0 \leqq s \leqq \delta; \tag{20}$$

(ii) the test function $V = \phi(f)$ for a strictly increasing, convex function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$.

(iii) for some $c, d < \infty$,

$$\mathscr{A}_m f(x) \leqq cf(x) + d, \qquad x \in O_m, \quad m \in \mathbb{Z}_+.$$

Then $\boldsymbol{\Phi}$ is ergodic and letting π denote its unique invariant probability we have

$$\lim_{t \to \infty} \|P^t(x, \cdot) - \pi\|_f = 0, \qquad x \in \boldsymbol{X}.$$

Proof. The first result follows from the argument used in Theorem 4.2 together with Theorem 6.3 of [25]. To prove (ii) we apply (CD2) and the comparison theorem to obtain the bound $P^sV \leqq V + ds$. Since ϕ is strictly increasing, its right derivative is bounded from below on $[1, \infty)$. From this fact and Jensen's inequality we may find a finite constant b such that

$$P^sf \leqq \phi^{-1}(\phi(f) + ds) \leqq f + bds, \qquad s \in \mathbb{R}_+$$

which shows that (20) is satisfied. The second result thus follows from (i).

To see that the third condition also suffices, note that since $f \geqq 1$ we may assume without loss of generality that $d = 0$. By Dynkin's formula applied to the function $\exp(-ct)f$ together with the now standard use of Fatou's lemma we obtain the bound

$$P^tf \leqq e^{ct}f, \qquad t \in \mathbb{R}_+.$$

This bound together with (i) immediately implies the result.

Theorem 5.3(i) leaves open the question of how Equation (20) may be verified. One method is by establishing (ii). This relationship between f and V is satisfied in many queueing models, where V is typically equal to f^q for some $q > 1$ (see the examples below and [21]). If this condition fails, then (iii) is also useful if the function f is sufficiently smooth.

Convergence in f-norm has recently been explored at some length in the case of discrete-time countable space by Hordijk and Spieksma [15], [29], and it is of considerable interest to have results which enable the stronger form of convergence to be used. Theorems 5.2–5.3 go some way in this direction: the general result is still an open question. When the convergence is exponentially fast, however, we can provide a full solution, and this is done in the next section.

6. Criteria for exponential ergodicity

Suppose that the Markov process $\boldsymbol{\Phi}$ is positive Harris recurrent with invariant measure π. For a function $f \geqq 1$ we say that $\boldsymbol{\Phi}$ is *f-exponentially ergodic* if there exists a constant $\beta < 1$ and a finite-valued function $B(x)$ such that

$$\|P^t(x, \cdot) - \pi\|_f \leqq B(x)\beta^t \qquad \forall t \in \mathbb{R}_+, \quad x \in \boldsymbol{X}.$$

This form of ergodicity is studied in [31], where it is shown that under some continuity conditions (in t) on the semigroup P^t, exponential ergodicity of the process $\boldsymbol{\Phi}$ follows from the geometric ergodicity of the embedded skeletons or of the resolvent chains.

Here we substantially improve on the results of [31] and show that a strengthened version of (CD2) provides a criterion for an exponential rate of convergence in the f-norm ergodic theorem, without any of the side conditions in Theorem 5.3.

(CD3) *Condition for exponential ergodicity.* The function V is norm-like, and for some $c > 0$, $d < \infty$,

$$\mathscr{A}_m V(x) \leqq -cV(x) + d, \qquad x \in O_m.$$

Theorem 6.1. Supose that $\boldsymbol{\Phi}$ is a right process, and that all compact sets are petite for some skeleton chain. If (CD3) holds, then there exists $\beta < 1$ and $B < \infty$ such that

$$\|P^t(x, \cdot) - \pi\|_f \leqq Bf(x)\beta^t, \qquad t \in \mathbb{R}_+, \quad x \in \boldsymbol{X},$$

with $f = V + 1$.

Proof. To begin, observe that (CD3) trivially implies (CD0). Hence by Theorem 2.1 the process is non-explosive. We first apply the extended generator $\mathscr{A}_m$ to the function $g(x, t) \triangleq V(x) \exp(ct)$ to obtain from the product rule

$$\begin{aligned} \mathscr{A}_m g(x, t) &= \exp(ct)[\mathscr{A}_m V(x) + cV(x)] \\ &\leqq d \exp(ct). \end{aligned}$$

By Dynkin's formula and the same argument used in the proof of the comparison theorem it follows that for all $t \in \mathbb{R}_+$ and $x \in \boldsymbol{X}$,

$$\begin{aligned} \exp(ct)P^t V(x) = \boldsymbol{E}_x[g(\Phi_t, t)] &\leqq V(x) + \liminf_{m\to\infty} \boldsymbol{E}_x\left[\int_0^t \mathscr{A}_m g(\Phi_s, s)\, ds\right] \\ &\leqq V(x) + d\int_0^t \exp(cs)\, ds \\ &\leqq V(x) + (d/c)\exp(ct). \end{aligned}$$

Thus for any distribution a on $\mathbb{R}_+$ with $a(0) < 1$,

$$K_a V \leqq \lambda V + d/c \tag{21}$$

where $\lambda = \int_0^\infty \exp(-ct)a(dt) < 1$.

Under the present hypotheses, for some δ the δ-skeleton with transition law P^δ is irreducible and aperiodic, and all compact subsets of $\boldsymbol{X}$ are P^δ-petite. Choosing $K_a = P^\delta$ in (21), by Theorem 6.3 of [24] there exist $\rho < 1$ and $R < \infty$ such that

$$\|P^{n\delta}(x, \cdot) - \pi\|_f \leqq Rf(x)\rho^n, \qquad n \in \mathbb{Z}_+, \quad x \in \boldsymbol{X},$$

where $f(x) = V(x) + 1$.

For arbitrary $t \in \mathbb{R}_+$ we may write $t = n\delta + s$, with $s \in [0, \delta)$, to estimate

$$\begin{aligned} \|P^t(x, \cdot) - \pi\|_f &= \sup_{|g|\leqq f} \left| P^{n\delta+s}g - \int g\, d\pi \right| \\ &\leqq \int P^s(x, dy)\, \|P^{n\delta}(y, \cdot) - \pi\|_f \\ &\leqq R\rho^n P^s f(x) \\ &\leqq R\rho^n(V(x) + d/c + 1) \end{aligned}$$

where the last inequality follows from (21), this time by choosing $a = \delta_s$; and the result is proved.

Condition (CD3) is frequently satisfied under any conditions where (CD2) can be checked. It is clearly the most useful of our conditions, in that it not only provides positive Harris recurrence and even exponential ergodicity, but also provides *automatically* the bounding constant in the f-norm convergence.

7. Application: chains with countable state space

Before giving more specific examples, it is worthwhile making these results a little more explicit for countable spaces, since the technical conditions are then much less onerous and the results more transparently stated. Because any function on a discrete state space is continuous, it trivially follows that a Markov process whose state space is discrete is a T-process. When the derivative exists, the extended generator $\mathscr{A}_m$ for the stopped process may be represented by a matrix Q where $q_{ij} = p'_{ij}(0)$ for $i \in O_m$.

The approaches above enable us to provide stronger results than those currently available, even in discrete state space. We give just the exponential ergodicity result here, for ease of use and interpretation: the next theorem is a direct consequence of Theorem 6.1, and extends Theorem 3(ii) of [36], whilst also showing that the hypotheses of Theorem 1.18 of Chen [6] give much stronger conclusions than those he draws.

Theorem 7.1. Suppose that $\boldsymbol{X} = \mathbb{Z}_+$ and $\boldsymbol{\Phi}$ is irreducible. If there exists a function V such that $V(j) \to \infty$ as $j \to \infty$, satisfying for some $c, d < \infty$

$$\sum_j q_{ij} V(j) \leqq -cV(i) + d, \qquad i \in \boldsymbol{X},$$

then there exists $\beta < 1$ and $B < \infty$ such that

$$\|P^t(i, \cdot) - \pi\|_f \leqq Bf(i)\beta^t, \qquad t \in \mathbb{R}_+, \quad i \in \boldsymbol{X},$$

with $f = V + 1$.

8. Application: a controlled linear system

In Section 3.3 of [25] we considered the diffusion process

$$d\Phi_t = Y(\Phi_t)\, dt + \sum_{i=1}^m X_i(\Phi_t) \circ dB_t \tag{22}$$

whose generator is a second-order differential operator of the form

$$\mathscr{A} = Y + \tfrac{1}{2} \sum_{i=1}^m X_i^2. \tag{23}$$

Since the sample paths of a diffusion are continuous, it follows that for the stopped process, any function $V: \boldsymbol{X} \to \mathbb{R}_+$ with continuous first and second partial derivatives is in the domain of $\mathscr{A}_m$ and on O_m,

$$\mathscr{A}_m V = YV + \tfrac{1}{2} \sum_{i=1}^{m} X_i^2 V.$$

Using this identity we now apply the results of the paper to obtain new results for a classical problem in the theory of passive linear systems.

Let $\boldsymbol{X} = \mathbb{R}^n$, and define $\boldsymbol{\Phi}$ as the state process for a linear system under memoryless non-linear control, following [38], by the specific form of (22)

$$d\Phi_t = F\Phi_t\, dt - b\phi(c^{\mathsf{T}}\Phi_t)\, dt + G(\Phi_t)\, dw_t. \tag{24}$$

From (23) we see that for any V with continuous first and second partial derivatives, the extended generator associated with (24) may be written, for any truncated process $\boldsymbol{\Phi}^m$, as

$$\mathscr{A}_m V(x) = [Fx - b\phi(c^{\mathsf{T}}x)]^{\mathsf{T}} \frac{\partial V(x)}{\partial x} + \frac{1}{2} \sum_{i,j} [G(x)G(x)^{\mathsf{T}}]_{i,j} \frac{\partial^2 V(x)}{\partial x_i\, \partial x_j}. \tag{25}$$

We impose the following conditions on the parameters of (24) so that $\boldsymbol{\Phi}$ is a T-process, and the freely evolving system (with $w_t \equiv 0$) will be exponentially asymptotically stable. This fact enables us to construct a test function of the form (CD3), which will be shown to imply geometric ergodicity for the state process.

(NL1) The function G is C^∞, and there exists a constant B for which

$$0 < G(x)G(x)^{\mathsf{T}} \leqq BI, \qquad x \in \boldsymbol{X}.$$

(NL2) All the eigenvalues of F have negative real parts.

(NL3) The function $x\phi(x)$ is positive for all x sufficiently large; and ϕ is C^∞ with ϕ' bounded on $\mathbb{R}$.

(NL4) There exist two non-negative constants γ and κ such that $\gamma + \kappa > 0$ and

$$\operatorname{Re}\,(\gamma + i\omega\kappa)c^{\mathsf{T}}(i\omega I - F)^{-1}b > 0$$

for all real ω.

In [38] it is shown that $\boldsymbol{\Phi}$ is positive Harris recurrent under conditions related to (NL1)–(NL4); but we can actually say far more.

Theorem 8.1. If (NL1)–(NL4) hold then $\boldsymbol{\Phi}$ is geometrically ergodic. Letting π denote its invariant probability, we have for some $B < \infty$, $\beta < 1$,

$$\|P^t(x, \cdot) - \pi\|_f \leqq Bf(x)\beta^t, \qquad x \in \boldsymbol{X}, \quad t \in \mathbb{R}_+,$$

with $f(\cdot) = |\cdot|^2 + 1$.

Proof. From [25] we have that by (NL1) the skeleton P^Δ is its own irreducible continuous component for each $\Delta > 0$. In [38] the function

$$V(x) = x^\top Px + \kappa \int_0^{c^\top x} \phi(s)\, ds$$

is considered, and by appropriate choice of $P > 0$ and $Q > 0$ it is shown under (NL2)–(NL4) that

$$[Fx - b\phi(c^\top x)]^\top \frac{\partial V(x)}{\partial x} \leqq -x^\top Qx, \qquad x \in \boldsymbol{X}$$

and

$$\tfrac{1}{2} \sum_{i,j} [G(x)G(x)^\top]_{i,j} \frac{\partial^2 V(x)}{\partial x_i\, \partial x_j} = \operatorname{trace}\, [G(x)G(x)^\top P] + \tfrac{1}{2}\kappa\, |G(x)^\top c|^2\, \phi'(c^\top x).$$

By (25), (NL1), and (NL3) we have for some $c > 0$, $d < \infty$,

$$\mathscr{A}_m V(x) \leqq -x^\top Qx + L \leqq -cV(x) + d.$$

Hence the geometric ergodicity result follows as an application of Theorem 6.1.

9. Application: storage processes with general release rule

Many common processes in continuous time occur in the operations research area, and we deal with a number of them below. Processes much more general than those we consider on a general space are shown to be right processes in [8], [9], and our results will in general hold for such processes: we do not pursue the details here.

In [13], [14] Harrison and Resnick construct infinitesimal generators for storage processes with compound Poisson input and a general deterministic release path between jumps of the compound Poisson process. In this section we apply the methods above to develop criteria for recurrence, ergodicity and geometric ergodicity of such storage processes. Let

$$\Phi_t = x + A(t) - \int_0^t r(\Phi_s)\, ds, \qquad t \geqq 0 \tag{26}$$

denote a general storage process as in [13]. Here $\{A(t), t \geqq 0\}$ is a compound Poisson process with rate λ and jump size distribution $H(\cdot)$, not degenerate at zero. As in [13], we assume the release function $r(\cdot)$ is strictly positive, left continuous and has a positive right limit everywhere in $(0, \infty)$. We take $r(0) = 0$. We assume further that for one and hence all $a > 0$

$$0 < R(a) \triangleq \int_0^a [r(u)]^{-1}\, du < \infty. \tag{27}$$

It was shown in Theorem 4.2 of [25] that under (27) the K_a-chain is an irreducible T-process for any a with $a(0) < 1$. The fact that, in particular, skeletons are T-processes is exploited in [32] to analyze ergodic properties. Our analysis below is both simpler and gives more detailed results, since we can use the generator directly.

In Proposition 4 of [13] and its corollary, the weak infinitesimal generator $\tilde{\mathscr{A}}$ of $\boldsymbol{\Phi}$ is shown to satisfy

$$\tilde{\mathscr{A}}V(x) = \lambda \int_0^\infty [V(x+y) - V(x)]H(dy) - r(x)V'(x) \tag{28}$$

where the domain of $\tilde{\mathscr{A}}$ consists of all bounded absolutely continuous functions V with left-continuous density V' such that $r(\cdot)V'(\cdot)$ is bounded on $(0, \infty)$.

Similarly, if we consider the expanding sets $O_m = [0, m)$ and the truncation $\boldsymbol{\Phi}^m$ to be stopped at m when $\boldsymbol{\Phi}$ leaves O_m, then as in the construction in [13], we get

$$\begin{aligned} \mathscr{A}_m V(x) = \lambda \int_0^m & [V(x+y) - V(x)]H(dy) \\ & + [V(x+m) - V(x)][1 - H(m)] - r(x)V'(x). \end{aligned} \tag{29}$$

It is then straightforward to see that for any increasing, unbounded V with a left-continuous density V' for which $\tilde{\mathscr{A}}V$ defined by (28) exists, V is in the domain of the extended generator of the stopped process, and $\mathscr{A}_m V(x) \leqq \tilde{\mathscr{A}}V(x)$ for all x, even though V may not be in the domain of $\tilde{\mathscr{A}}$ as a generator of $\boldsymbol{\Phi}$.

Thus we can use (29) to develop a test-function approach as follows.

Theorem 9.1. Assume $R(x) \to \infty$ as $x \to \infty$, and write

$$L(x) = \lambda \int_0^\infty \int_x^{x+y} [r(u)]^{-1}\, du H(dy). \tag{30}$$

(i) If $L(x) \leqq 1$ for all $x \geqq x_0$, then $\boldsymbol{\Phi}$ is Harris recurrent.

(ii) Suppose that for $x \geqq x_0$, $L(x) \leqq 1 - \varepsilon$ for some $\varepsilon > 0$; then $\boldsymbol{\Phi}$ is Harris ergodic, and

$$\|P^t(x, \cdot) - \pi\| \to 0$$

for all starting points x.

(iii) Assume that for all $x \geqq x_0$, $L(x) \leqq 1 - \varepsilon$, and for some $c^* > 0$, $D < \infty$,

$$\sup_{x \geqq 0} \int_0^\infty \exp\left[c^* \int_x^{x+y} [r(u)]^{-1}\, du\right] H(dy) = D < \infty. \tag{31}$$

Then $\boldsymbol{\Phi}$ is geometrically ergodic, and for some constants $B < \infty$, $\beta < 1$

$$\|P^t(x, \cdot) - \pi\|_f \leqq Bf(x)\beta^t$$

with $f(\cdot) = \exp(\alpha R(\cdot))$.

Proof. (i) We use $V(x)=R(x)$ in (29), and have that for $x \geqq x_0$

$$\mathscr{A}_m R(x) \leqq \lambda \int_0^\infty \left(\int_x^{x+y} [r(u)]^{-1}\, du \right) H(dy) - 1 \leqq 0;$$

since $R(x) \to \infty$, $x \to \infty$, the result follows from Theorem 3.3.

(ii) From Theorem 5.1 it is enough to prove that ($\mathscr{S}$) holds. But exactly as in (i), if $L(x) \leqq 1 - \varepsilon f(x)$, we have (CD2) holding for each m, and $x \geqq x_0$. To see that drift away from the origin is controlled, note that we have R bounded on compacta since it is continuous. Moreover, for $x \leqq x_0$

$$\begin{aligned} \mathscr{A}_m R(x) &\leqq \lambda \int_0^\infty \left(\int_x^{x+y} [r(u)]^{-1}\, du \right) H(dy) - 1 \\ &\leqq \lambda \int_0^\infty \left[\int_0^{x_0} [r(u)]^{-1}\, du + \int_{x_0}^{x_0+y} [r(u)]^{-1}\, du \right] H(dy) - 1 \qquad (32) \\ &\leqq \lambda R(x_0) - \varepsilon, \end{aligned}$$

and so we have Harris ergodicity from Theorem 5.1.

(iii) Choose the test function $V_c(x) = \exp(cR(x))$, where $c > 0$ is to be selected later. Then from (29)

$$\begin{aligned} \mathscr{A}_m V_c(x) &\leqq \lambda \int_0^\infty [\exp(cR(x+y)) - \exp(cR(x))] H(dy) - c \exp(cR(x)) \\ &= cV_c(x) \left\{ \lambda \int_0^\infty c^{-1} [\exp(c[R(x+y) - R(x)]) - 1] H(dy) - 1 \right\}. \end{aligned}$$

From Taylor's theorem we have

$$\begin{aligned} \exp(c[R(x+y) - R(x)]) = 1 &+ c[R(x+y) - R(x)] \\ &+ (c^2/2)[R(x+y) - R(x)] \exp(\xi) \end{aligned} \qquad (33)$$

where $\xi \leqq c[R(x+y) - R(x)]$. Using this upper bound for ξ in (33), we have

$$\begin{aligned} \mathscr{A}_m V_c \leqq cV_c(x) \Big\{ &\lambda \int_0^\infty [R(x+y) - R(x) - 1/\lambda] H(dy) \\ &+ \frac{c\lambda}{2} \int_0^\infty [R(x+y) - R(x)]^2 \exp(c[R(x+y) - R(x)]) H(dy) \Big\}. \end{aligned} \qquad (34)$$

Now for $x \geqq x_0$, as in (ii) the first integrand in (34) is less than $-\varepsilon$, and is bounded for all $x \leqq x_0$ as in (32). Using the simple bound $x^2 \leqq 2\Delta^{-2} \exp(\Delta x)$, and choosing $\Delta = c^*/2$, $c < \Delta$, we have the second integral in (34) bounded by

$$2\Delta^{-2} \int_0^\infty \exp(c^*[R(x+y) - R(x)]) H(dy) \leqq 2\Delta^{-2} B$$

if $x \geqq x_0$, and bounded as in (32) for $x \leqq x_0$. Now choose $0 \leqq c < \varepsilon \Delta^2 (2B\lambda)^{-1}$. Then in (34), for $x \geqq x_0$,

$$\mathscr{A}_m V_c(x) \leqq -(c\varepsilon/2)V_c(x), \qquad x \geqq x_0,$$

and $\mathscr{A}_m V_c(x)$ is bounded, $x \leqq x_0$. The required result now follows from Theorem 6.1.

In most cases in the literature, $r(x) \to \infty$ as $x \to \infty$; cf [7], [2], [3]. It is possible to assume the weaker condition, covering these practical cases, that for some $\gamma > 0$

$$\liminf_{x \to \infty} r(x) \geqq \gamma. \tag{35}$$

The case with $r(u)$ bounded from zero in this way simplifies the situation. If (27) holds, we have $R(x) \to \infty$ as $x \to \infty$ and our recurrence result (i) needs no extra conditions. Moreover, in this situation we can replace the condition (31) with the single uniform condition

$$\int_0^\infty \exp(c^* y) H(dy) < \infty, \qquad \text{some } c^* > 0. \tag{36}$$

In [3], Section 7, it is shown that (ii) implies positive recurrence for general Levy process input: the proof uses notably deeper properties of the storage process structure. In [14], the conditions for Harris recurrence are not explicit, and our result (i) seems to be new, if not unexpected. There is a version of our geometric ergodicity result in [32], showing (36) to be sufficient for geometric ergodicity, for the constant release rate model. The bound on (iii) and the use of the f-norm is of course new.

We note that, as in Section 7 of [3], the ergodicity conditions of (ii) are necessary and sufficient if r is monotone increasing.

10. Application: work-modulated queues

In [4] the ergodic properties of two forms of work-modulated queueing models are considered. In one of these (their Model 2), the work rate of the server depends on the amount of work in the system. This model is similar to that in the previous section, and can be analyzed similarly although in [4] the arrival rate of customers is also allowed to depend on the work in the system. Here we describe Model 1 introduced in [4] to handle situations where the service times of customers entering the queue also depend on the state of the system as the customers arrive.

Consider a single-server queue where the nth customer arrives at time t_n, $n \geqq 1$. We take $t_0 = 0$. We let $W(t)$ denote the work, measured in units of time, which is in the system at time t. Hence at time t_n, the nth customer will experience a delay of $W(t_n-)$ before entering service. At time t_n, customer n is allotted a service time S_n which depends upon the past of the system as follows:

$$\boldsymbol{P}\{S_n \in A \mid S_0, \cdots, S_{n-1}, W_t : t < t_n\} = H(W(t_n-), A)$$

where H is a Markovian kernel on $\mathbb{R}_+ \times \mathcal{B}(\mathbb{R}_+)$. The arrival times $\{t_n : n \in \mathbb{Z}_+\}$ form a non-stationary Poisson process with intensity $\lambda(W(t))$, also depending on the work in the system, where λ is a bounded, non-negative function with $\lambda(0) > 0$. We assume that

$$\boldsymbol{P}\{\text{an arrival occurs in } [t, t+h] \mid W(s) : s \leqq t\} = \lambda(W(t))h + o(h).$$

Davis [9] shows that such a Markov process is a right process on $\mathbb{R}_+$. Since λ is bounded it may easily be seen that the process is non-explosive; while the fact that this queueing model is an irreducible T-process follows exactly as in Theorem 4.2 of [25] for storage models. To obtain (6) for suitable functions V we will assume that λ is left continuous, and that the transition function H has the Feller property.

We will use this example to illustrate how the methods introduced in this paper may be used to establish (i) recurrence and ergodicity; (ii) finiteness of moments of the steady-state work in the system; and (iii) exponential convergence of the expected work, and other functions of the state process. We first need to develop the form of the extended generator for the work-modulated queue.

Let $O_m = [0, m)$, $m \in \mathbb{Z}_+$, and let $\mathcal{A}_m$ be the extended generator for the process $\boldsymbol{W}^m$ defined by $W^m_t = m$ for $t \geqq T^m$. For an initial condition $x \in O_m$, we have $T^m \triangleq \tau_{O^c_m} > t$ on the event {no arrivals occur in $[0, t]$}. From this fact and the basic definitions we obtain for $x > t > 0$,

$$\begin{aligned}\boldsymbol{E}_x[V(W^m_t)] = {} & V(x-t)(1 - t\lambda(x)) \\ & + t\lambda(x)\int_{s \in [0,t]} \left\{ \int_{y \in [0, m-x+s)} V(x-t+y)H(x, dy) \right. \\ & \left. + V(m)H(x, [m-x+s, \infty)) \right\} \boldsymbol{P}\{t_1 \in ds \mid t_1 \leqq t\} + o(t).\end{aligned}$$

Rearranging terms gives

$$\begin{aligned}\frac{\boldsymbol{E}_x[V(W^m_t)] - V(x)}{t} = {} & \frac{V(x-t) - V(x)}{t} - \lambda(x)V(x-t) \\ & + \lambda(x)\int_0^\infty V((x-t+y) \wedge m)H(x, dy) + o(1).\end{aligned}$$

This equation also holds for $x = 0$ on replacing $V(x-t)$ by $V(0)$. For differentiable $V : \mathbb{R}_+ \to \mathbb{R}$ we obtain the identity, for arbitrary $x \in \mathbb{R}_+$,

$$\begin{aligned}\lim_{t \downarrow 0} \frac{\boldsymbol{E}_x[V(W^m_t)] - V(x)}{t} = {} & -V'(x)\boldsymbol{1}\{x > 0\} \\ & + \lambda(x)\int_0^\infty (V((x+y) \wedge m) - V(x))H(x, dy).\end{aligned}$$

Our assumption that λ is left continuous and H has the Feller property ensures that the domain of $\mathcal{A}_m$ contains $C^1(\mathbb{R}_+)$ for any finite m. Finally, for an increasing, positive function $V \in C^1(\mathbb{R}_+)$ we have the bound

$$\mathcal{A}_m V(x) \leqq -V'(x)\mathbf{1}\{x>0\} + \lambda(x)\int_0^\infty (V(x+y)-V(x))H(x,dy). \tag{37}$$

With this bound we may apply the results of this paper to obtain rates of convergence of the process $\boldsymbol{W}$ to a stationary regime, and bounds on moments of the steady state queue size.

(i) *Recurrence and positive recurrence.* Consider the test function $V(x)=x$. We have seen that V is in the domain of the extended generator for the stopped process, and

$$\mathcal{A}_m V(x) \leqq -1 + \lambda(x)h(x), \qquad x>0, \quad m \in \mathbb{Z}_+ \tag{38}$$

where $h(x) \triangleq \int yH(x,dy)$, $x \in \mathbb{R}_+$. Applying Theorem 3.2 we obtain the following result.

Theorem 10.1. If $\lambda(x)h(x) \leqq 1$ for all x sufficiently large, then the process $\boldsymbol{W}$ is Harris recurrent.

To obtain positivity this condition must be strengthened slightly in the usual way. The next result follows from (38) and Theorem 4.2.

Theorem 10.2. Assume $\limsup_{x\to\infty} \lambda(x)h(x) < 1$; then the process $\boldsymbol{W}$ is positive Harris recurrent.

(ii) *Finiteness of moments.* Here we consider the test function $V(x)=x^n$ to obtain bounds on the $(n-1)$th moments of the steady state work in the queue. For this we assume that $\int y^n H(x,dy)$ is uniformly bounded in x. From (37) we have, for $x>0$,

$$\begin{aligned}\mathcal{A}_m V(x) &\leqq -nx^{n-1} + \lambda(x)\int \left(\sum_{i=1}^n \binom{n}{i} x^{n-i}y^i\right)H(x,dy)\\ &= -nx^{n-1}(1-\lambda(x)h(x)) + O(x^{n-2}).\end{aligned}$$

This and Theorem 5.3(ii) immediately gives the following result.

Theorem 10.3. Assume $\limsup_{x\to\infty}\lambda(x)h(x)<1$ and $\sup_{x\geqq 0}\int y^nH(x,dy)<\infty$ for some fixed n. Then the process $\boldsymbol{W}$ is positive Harris recurrent with invariant probability π, $\boldsymbol{E}_\pi[W_0^{n-1}]<\infty$, and

$$\lim_{t\to\infty} \|P^t(x,\cdot)-\pi\|_f \to 0, \qquad x \in \boldsymbol{X},$$

with $f(\cdot) \triangleq |\cdot|^{n-1}+1$.

(iii) *Exponential ergodicity.* To obtain exponential ergodicity we set $V(x)=\exp(\alpha x)$, and apply (37) to obtain the bound

$$\mathcal{A}_m V(x) \leqq -\alpha V(x) + \lambda(x)V(x)\int (V(y)-1)H(x,dy). \tag{39}$$

Theorem 10.4. Assume $\limsup_{x\to\infty} \lambda(x)h(x) < 1$ and $\sup_{x\geq 0} \int \exp(\beta y) H(x, dy) < \infty$ for some $\beta > 0$. Then the process $\boldsymbol{W}$ is positive Harris recurrent with invariant probability π such that

$$\int \pi(dy) \exp(\alpha y) < \infty$$

for some $\alpha > 0$. Moreover $\boldsymbol{W}$ is exponentially ergodic, and there exists $\beta < 1$, $B < \infty$ such that

$$\|P^t(x, \cdot) - \pi\|_f \leqq Bf(x)\beta^t$$

with $f(\cdot) = \exp \alpha\, |\cdot|$.

Proof. By Taylor's theorem we have the bound, valid for any $y \in \mathbb{R}_+$,

$$V(y) \leqq 1 + \alpha y + \frac{\alpha^2 y^2 V(y)}{2}.$$

Hence from (39) and the hypotheses of the theorem we have, for some $M < \infty$ and $\alpha > 0$ sufficiently small,

$$\mathcal{A}_m V(x) \leqq -\alpha(1 - \lambda(x)h(x))V(x) + \alpha^2 M \lambda(x) V(x).$$

Since λ is bounded, we see that the drift condition (CD3) is satisfied for all α sufficiently small. This with Theorem 6.1 completes the proof.

11. Application: risk models

In [14], a risk model with instantaneous increase at rate $r(\Phi_s)$ between the (downward) jumps of a Poisson process is defined. Assume r is continuous with $R(\infty) = \infty$, and that $\{A(t), t \geqq 0\}$ and r are as in Section 9. Then Proposition 4 of [14], and the construction (16) there show the generator of the risk process $\boldsymbol{\Phi}$ is given by

$$\begin{aligned} \mathcal{A}V(x) = r(x)V'(x) - \lambda \int_0^x [V(x) - V(x-y)] H(dy) \\ -\lambda H(x, \infty)[V(x) - V(0)], \end{aligned} \tag{40}$$

with domain containing those bounded continuous V for which $r(x)V'(x)$ is bounded and continuous, and has a finite limit as $x \to 0$. Note from the construction in [14] that, because of the continuous paths upwards $\mathcal{A}_m$ has a form identical to $\mathcal{A}$, on $O_m = [0, m)$, provided we place $\boldsymbol{\Phi}$ at m when the process leaves O_m. Thus we can again relax the boundedness assumption for increasing functions V by considering $\mathcal{A}_m$.

Theorem 11.1. Define $J : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$J(x) = \lambda \int_0^x H(x-y, \infty)[r(y)]^{-1}\, dy. \tag{41}$$

(i) Suppose $J(x) \geqq 1$ for $x \geqq x_0$. Then the risk process is Harris recurrent.

(ii) Suppose $J(x) \geqq 1+\varepsilon$, $\varepsilon > 0$, for $x \geqq x_0$. Then the risk process is Harris ergodic.

(iii) Suppose that, for $x \geqq x_1$, and $\varepsilon > 0$, $B < \infty$,

$$\lambda \int_0^x \int_{x-y}^x [r(u)]^{-1}\, du H(dy) \geqq 1+\varepsilon \tag{42}$$

$$\lambda \int_0^x \left[\int_{x-y}^x [r(u)]^{-1}\, du\right]^2 H(dy) \leqq B. \tag{43}$$

Then the risk process is geometrically ergodic.

Proof. We first sketch the proof that Φ_Δ is a T-process. For any x, let η_x denote the point such that $P_x(\Phi_\Delta = \eta_x \mid A_\Delta = 0) = 1$. Suppose the support of H is unbounded. Then by considering only paths for which the first jump occurs in $[0, \Delta]$, we have $P^\Delta(x, B) \geqq \int_0^\Delta \lambda[1-H(\eta_x)]P^{\Delta-s}(\{0\}, B)$, and this provides an appropriate component if H is continuous. If H has bounded support, then an iterative version of this construction is needed, whilst if H is discontinuous then a smoothed version can be used: we omit the details. The Harris recurrence and Harris ergodicity conditions then follow by noting that the right-hand side of (40), with $V(x) = R(x)$, is given by $1 - J(s)$ as in (16) of [14], whilst clearly $\mathscr{A}R(x)$ is always bounded.

The geometric ergodicity is a trifle more delicate. We first note that if (42) holds, then (40) with $V = R$ again implies Harris ergodicity. Now fix the test function $V_c(x) = \exp(cR(x))$. Then from (40)

$$\begin{aligned} \mathscr{A}_m V_c(x) &\leqq c \exp(cR(x)) - \lambda \int_0^x [\exp(cR(x)) - \exp(cR(x-y))]H(dy) \\ &\leqq c \exp(cR(x))\left[1 - (\lambda/c)\int_0^x [1 - \exp(c(R(x) - R(x-y)))\right]H(dy). \end{aligned} \tag{44}$$

The integration (44) can be expanded as in (33) to see, for some $\xi_y \leqq c[R(x) - R(x-y)]$,

$$\begin{aligned} &\lambda c^{-1} \int_0^x [1 - \exp(-c(R(x) - R(x-y)))]H(dy) \\ &= \lambda \int_0^x [R(x) - R(x-y)]H(dy) \\ &\quad - \frac{\lambda c}{2}\int_0^x (R(x) - R(x-y))^2 \exp(-c\xi_y)H(dy) \\ &\geqq 1 + \varepsilon/2, \end{aligned} \tag{45}$$

provided c is chosen small enough, from (43), that

$$\lambda c \int_0^x [R(x) - R(x-y)]^2 H(dy) < \varepsilon. \tag{46}$$

When (46) holds, from (45) and (44) we have

$$\mathcal{A}_m V_c(x) \leqq -(\varepsilon c/2) V_c(x), \qquad x \geqq x_0.$$

Since $\mathcal{A}_m V_c(x) \leqq c \exp(cR(x_0))$, $x \leqq x_0$, the result follows.

The geometric ergodicity condition (42) is stronger in general than the condition $J(x) \geqq 1+\varepsilon$ in (ii). However, when $r(u) \equiv r$, it is easy to see that they are equivalent, since $J(x) \geqq 1+\varepsilon$ entails, for all $x \geqq x_0$

$$\lambda \int_0^x yH(dy) + \lambda x H(x, \infty) \geqq r(1+\varepsilon); \tag{47}$$

and so if $\lambda \int_0^\infty yH(dy) \leqq r$ (so that (42) fails), then since $\lambda x H(x, \infty) \leqq \lambda \int_x^\infty yH(dy) \to 0$ as $x \to \infty$, we cannot have (47).

In this case of constant increase, the condition (43) is equivalent to

$$\lambda / r^2 \int_0^\infty y^2 H(dy) < \infty. \tag{48}$$

Intuitively, this says that all ergodic risk processes are in fact geometrically ergodic provided that the downward 'drift' provided by the compound Poisson process is not due to rare, but very large, jumps. We conjecture that (43) is not necessary for geometric ergodicity, and that a more delicate choice of test function may enable it to be weakened.

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