Continuum limit

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1 Functional forms

The continuum limit is a critical step in any precision lattice calculation. In this study, we take advantage of the symmetries of the reduced pseudo-ITD to parameterize the lattice spacing correction.

The matrix element has the property

$$\mathfrak{M}(p,z,a) = \mathfrak{M}^*(-p,z,a) = \mathfrak{M}^*(p,-z,a) = \mathfrak{M}(-p,-z,a)\,, \tag{1}$$

which we used when constructing the summed 3pt correlation functions to increase statistical precision. The last relation restricts lattice spacing errors with odd powers must be functions of $a|p|$ and $a/|z|$.

A Taylor expansion in lattice spacing will give the expansion

$$\mathfrak{M}(p,z,a) = \mathfrak{M}_{\rm cont}(\nu,z^2) + \sum_n (a|p|)^n P_n(\nu) + (\frac{a}{|z|})^n Q_n(\nu)\,. \tag{2}$$

It is important to note that the coefficients of the lattice spacing errors can be functions of the Ioffe time. A previous parameterization of lattice spacing errors for parton observables have neglected this potentially significant feature [?]. In fact, the leading term, independent of Ioffe time, which they do include is not present in this calculation. It is cancelled in the ratio defining the reduced pseudo-ITD, just like the leading higher twist effects. In the recent work [?], the Ioffe time dependence is taken into account by fitting the lattice spacing dependence by fixing p and fitting an interpolation of their data in z. In this study, we do not have ensembles with different lattice spacing and the same spatial extent, so this simpler technique cannot be used.

As will be seen, the terms higher order in Ioffe time are necessary for a reliable continuum limit since all lattice calculations of partonic distributions utilize $\nu > 1$. There are also potential z^2 dependence on the lattice spacing coefficients. Those effects would be suppressed either by α or $\Lambda^2_{\rm QCD} z^2$, so they will be neglected here.

Using the relationships in Eq. 1, we can see that the terms in P_n and Q_n which are odd in ν are purely imaginary and the terms even in ν are purely real, just like reduced pseudo-ITD. Since we will be working in a fixed frame and only have access to data with positive p, z, and ν, this parameterization can be rearranged by remembering $|\nu| = |z||p|$. This relation allows for mixing terms from P_n and Q_n. One could replace $(\frac{a}{|z|})^n Q_n(\nu)$ with a $(a|p|)^n P'_n(\nu)$ where $P'(\nu)$ would have real odd powers of $|\nu|$ terms and imaginary even powers of $|\nu|$. For this study, we will make this substitution and fit ap errors for all powers of $|\nu|$ up to $n_{\rm ap}$.

Even though the action is $O(a)$ improved, the Wilson line operator is not, so we will work solely with $O(a)$ errors. Since there appears to be no large lattice spacing dependence in Fig. **??**, we expect higher orders in a will be small.

For this study, we will parameterize P_1 as the sum of Chebyshev polynomials, $T_n(x)$,

$$P_1(\nu) \sim \sum_{n=1}^{n_{\rm ap}} A_n^{\rm (ap)} T_n(\frac{\nu}{\nu_{\rm max}}) \tag{3}$$

where $\nu_{\rm max}$ is the largest Ioffe time used in the analysis and $A_n^{\rm (ap)}$ are complex fit parameters. Similarly Q_1 is defined where the sum goes up to $n_{\rm az}$ and the $A_n^{\rm (az)}$ are fit. We will also parameterize the continuum reduced pseudo-ITD as

$$\mathfrak{M}_{\rm cont}(\nu, z^2) \sim \sum_{n=0}^{n_{\rm lt}} a_n(\mu^2) c_n(\mu^2 z^2) \frac{(i\nu)^n}{n!} + z^2 \sum_{n=1}^{n_{\rm ht}} i^n B_n T_n(\frac{\nu}{\nu_{\rm max}}) , \tag{4}$$

where $c_n(\mu^2 z^2)$ is the perturbative matching kernel for the $n^{\rm th}$ moment, $a_n(\mu^2)$. The higher twist nuisance parameters B_n will be fit parameters. The continuum limit moments will be determined from model PDF of the form

$$f(x) = \frac{x^a(1-x)^b(1 + \sum_n^{n_{\rm pdf}} d_n x^{n/2})}{B(a+1, b+1) + \sum_n^{n_{\rm pdf}} d_n B(a+1+n/2, b+1)} , \tag{5}$$

in analogy to the procedure performed by phenomenological determinations of the PDF.

2 Fits

I have found the best fit for a wide range of functional forms with the models described in Table 1 In

$n_{\rm lt}$	6, 8, 10
$n_{\rm ht}$	0...8
$n_{\rm ap}$	0...8
$n_{\rm pdf}$	2, 4, 6

Table 1: The parameters defining the models.

general I have found that the χ^2/dof was consistently smaller for the real component, almost always being less than 2. The imaginary component was much more sensitive to fit models. For example for a $\chi^2/{\rm dof} < 4$, the $n_{\rm ap}$ had to be at least 2. Fig 1 shows ITDs for the best fit with the lowest average χ^2/dof for the real component, model 8, 4, 7, 2, and for the imaginary component, model 6, 4, 7, 2. Fig 2 show the nuisance polynomials. Fig 3 shows the PDFs. It should be noted that these average χ^2/dof are well within statistical errors of other models. It wouldn't be appropriate to just consider this model as the best.

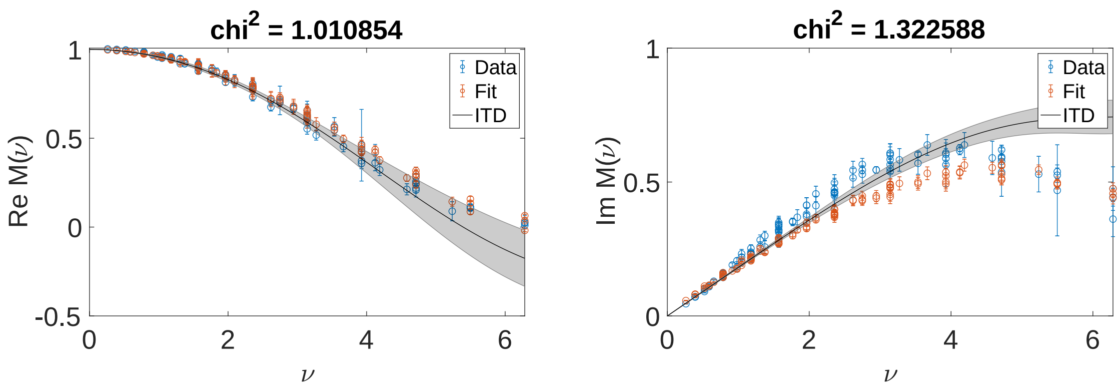

Figure 1: The best fit ITDs.

Instead of being reliant on the best fit, we can use some sophisticated statistical methods for combining these models. The recent MSU work used a technique of creating a weighted average using the AIC. The AIC is defined by

$$A = \chi^2 - 2k \tag{6}$$

where k is the number of parameters in a model. The weights for the average are

$$w_i = e^{-A_i/2} / \sum_j e^{-A_j/2} \tag{7}$$

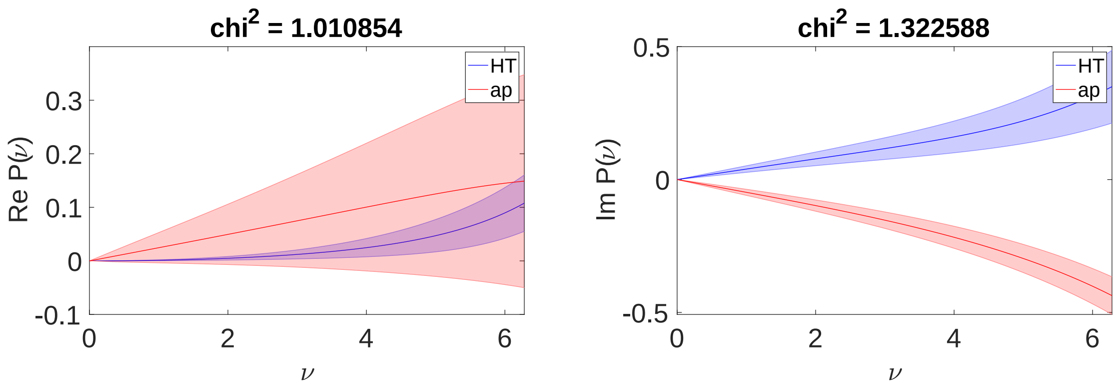

Figure 2: The best fit nuisance polynomials.

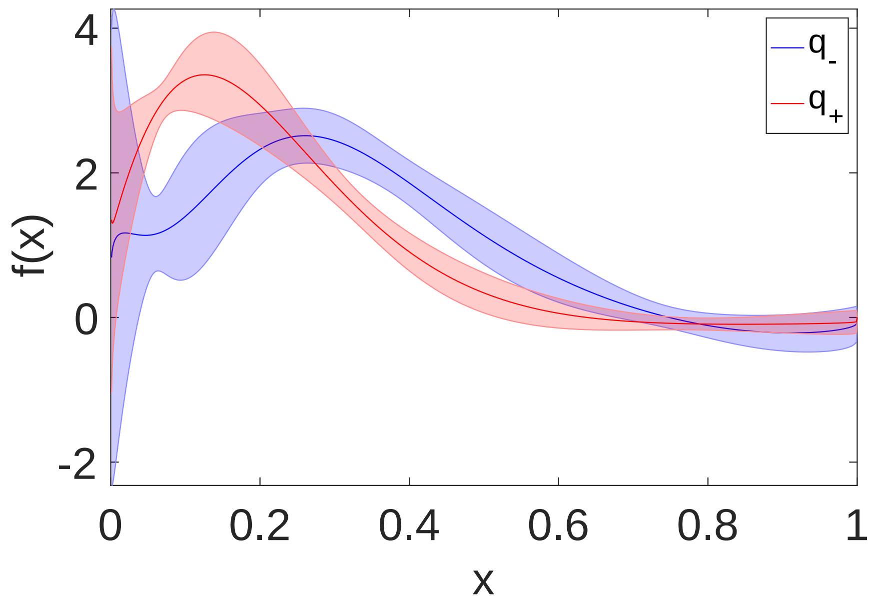

Figure 3: The best fit PDFs.

for the i^{th} model. I have performed this model average over all of the fits I've done. Unfortunately the PDF models with $n_{\text{pdf}} > 2$ are extremely noisy in the real component. It appears they have lost the strong correlations in the non exponent parameters, leading to wild fluctuations. Fig 4 shows the AIC weighted average for all models and for all models with $n_{\text{pdf}} = 2$.

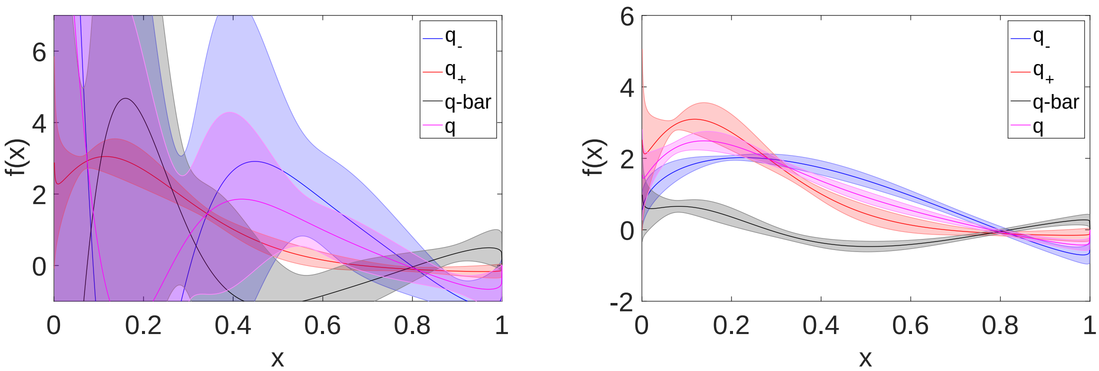

Figure 4: The AIC weighted average for all models (left) and for all models with $n_{\text{pdf}} = 2$ (right).