Notes on the ratio method for gluons
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We need to examine the reduced Ioffe time distribution

$$\mathcal{M}(\nu, z^2) = \frac{M_g(\nu, z^2)}{M_g(0, z^2)}, \tag{1}$$

where M_g is defined by

$$M_g(\nu, z^2) = \frac{1}{E^2}\langle P|\left[\mathcal{O}^{tjtj}(z,0) + \mathcal{O}^{ijij}(z,0)\right]|P\rangle, \tag{2}$$

The gluon matrix element renormalises multiplicatively, so the UV divergences should cancel, just as they do for the quark case. In fact, the above discussion goes through, as it does for the quark case, although there is one complication. The denominator in the light-cone limit for this case is

$$M_g(0,0) = \frac{1}{M_N^2}\langle M_N|\left[\mathcal{O}^{tjtj}(0,0) + \mathcal{O}^{ijij}(0,0)\right]|M_N\rangle = \int_0^1 \mathrm{d}x\, x f_g(x,\mu^2) \equiv \langle x\rangle(\mu^2), \tag{3}$$

which is no longer simply one and must be determined nonperturbatively.

The reduced matrix element is now

$$\begin{aligned}
\mathcal{M}_g(\nu, z^2) &= \frac{\int \mathrm{d}u\, Z_g(u, z^2\mu^2) M_g(u\nu, 0, \mu^2)}{M_g(0, z^2, \mu^2)} + \mathcal{O}(z^2) \\
&= \frac{\int \mathrm{d}u\, Z_g(u, z^2\mu^2) M_g(u\nu, 0, \mu^2)}{\int \mathrm{d}v\, Z_g(v, z^2\mu^2) M_g(0, 0, \mu^2)} + \mathcal{O}(z^2) \\
&= \frac{\int \mathrm{d}u\, Z_g(u, z^2\mu^2) M_g(u\nu, 0, \mu^2)}{\langle x\rangle(\mu^2) \int \mathrm{d}v\, Z_g(v, z^2\mu^2)} + \mathcal{O}(z^2) \\
&= \frac{\int \mathrm{d}u\, Z_g(u, z^2\mu^2) \mathcal{M}_g(u\nu, 0, \mu^2)}{\int \mathrm{d}v\, Z_g(v, z^2\mu^2)} + \mathcal{O}(z^2) \\
&= C_g(z^2\mu^2) \int \mathrm{d}u\, Z_g(u, z^2\mu^2) \mathcal{M}_g(u\nu, 0, \mu^2) + \mathcal{O}(z^2) \\
&= \int \mathrm{d}u\, \overline{Z}_g(u, z^2\mu^2) \mathcal{M}_g(u\nu, 0, \mu^2) + \mathcal{O}(z^2) \\
&= \int \frac{\mathrm{d}u}{|\nu|} \overline{Z}_g\left(\frac{u}{\nu}, z^2\mu^2\right) \mathcal{M}_g(u, 0, \mu^2) + \mathcal{O}(z^2),
\end{aligned} \tag{4}$$

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where $\overline{Z}_g$ can be obtained order-by-order from Z_g and C_g and we have

$$\mathcal{M}_g(\nu, 0) = \frac{M_g(\nu, 0, \mu^2)}{\langle x\rangle(\mu^2)}. \tag{5}$$

The relation between the gluon PDF and the matrix element is now

$$f_{g/H}(x, \mu^2) = \int \frac{\mathrm{d}\nu}{\pi x} e^{-ix\nu} M_g(\nu, 0, \mu^2) + \mathcal{O}(z^2), \tag{6}$$

where the factor of $2/x$ has been absorbed into the usual $1/(2\pi)$ from the (inverse) Fourier transform[1]. Note that we calculate $\mathcal{M}_g(\nu, 0)$, but we need $M_g(\nu, 0, \mu^2$, so we should write

$$\frac{f_{g/H}(x, \mu^2)}{\langle x\rangle(\mu^2)} = \int \frac{\mathrm{d}\nu}{\pi x} e^{-ix\nu} \mathcal{M}_g(\nu, 0, \mu^2) + \mathcal{O}(z^2), \tag{7}$$

Thus

$$\begin{aligned}
\mathcal{M}_g(\nu, z^2) &= \int \frac{\mathrm{d}u}{|\nu|} \overline{Z}_g\left(\frac{u}{\nu}, z^2\mu^2\right) \frac{2}{\langle x\rangle(\mu^2)} \int \mathrm{d}x\, e^{ixu} x f_{g/H}(x, \mu^2) + \mathcal{O}(z^2) \\
&= \frac{2}{\langle x\rangle(\mu^2)} \int \mathrm{d}x \int \frac{\mathrm{d}u}{|\nu|}\, e^{ixu}\, \overline{Z}_g\left(\frac{u}{\nu}, z^2\mu^2\right) x f_{g/H}(x, \mu^2) + \mathcal{O}(z^2) \\
&= \frac{1}{\langle x\rangle(\mu^2)} \int \frac{\mathrm{d}x}{x} 2 \int \mathrm{d}v\, e^{iv\nu}\, \overline{Z}_g\left(\frac{v}{x}, z^2\mu^2\right) x f_{g/H}(x, \mu^2) + \mathcal{O}(z^2) \\
&= \frac{1}{\langle x\rangle(\mu^2)} \int \frac{\mathrm{d}x}{x}\, \overline{C}_g\left(\frac{\nu}{x}, z^2\mu^2\right) x f_{g/H}(x, \mu^2) + \mathcal{O}(z^2).
\end{aligned} \tag{8}$$

To collect together things that we calculate on the lattice, we can write this as

$$\langle x\rangle(\mu^2) \mathcal{M}_g(\nu, z^2) = \int \frac{\mathrm{d}x}{x}\, \overline{C}_g\left(\frac{\nu}{x}, z^2\mu^2\right) x f_{g/H}(x, \mu^2) + \mathcal{O}(z^2), \tag{9}$$

where

$$\begin{aligned}
\overline{C}_g\left(\nu, z^2\mu^2\right) &= 2\int \mathrm{d}v\, e^{iv\nu}\, \overline{Z}_g\left(v, z^2\mu^2\right) \\
&= 2C_g(z^2\mu^2) \int \mathrm{d}v\, e^{iv\nu}\, Z_g(v, \mu^2 z^2) \\
&= \frac{2}{\int \mathrm{d}u\, Z_g(u, \mu^2 z^2)} \int \mathrm{d}v\, e^{iv\nu}\, Z_g(v, \mu^2 z^2),
\end{aligned} \tag{10}$$

[1] The factor of two comes from the Lorentz decomposition of the general matrix element $M^{\mu\nu\rho\sigma}$ and the factor of x is always a thing for gluons.

and $Z_g(v, \mu^2 z^2)$ is obtained from

$$M_g(\nu, z^2, \mu^2) = \int \mathrm{d}u\, Z_g(u, \mu^2 z^2) M_g(u\nu, 0, \mu^2) + \mathcal{O}(z^2). \tag{11}$$

From this discussion, it is clear that we need the normalisation

$$M_g(0, 0, \mu^2) = \int_0^1 \mathrm{d}x\, x f_g(x, \mu^2) \equiv \langle x \rangle(\mu^2), \tag{12}$$

which must be determined nonperturbatively via $M_g(0, 0, \tau)$ (and matched to the $\overline{\text{MS}}$ scheme). This matching procedure is of course carried out perturbatively. So if we write[2]

$$\langle x \rangle^{\overline{\text{MS}}}(\mu^2) = \mathcal{F}(\mu^2 \tau) \langle x \rangle(\tau), \tag{13}$$

then we have

$$\langle x \rangle(\tau) \mathcal{M}_g(\nu, z^2) = \int \frac{\mathrm{d}x}{x} \widetilde{C}_g \left(\frac{\nu}{x}, z^2 \mu^2, \mu^2 \tau \right) x f_{g/H}(x, \mu^2) + \mathcal{O}(z^2), \tag{14}$$

with a left-hand side that is determined on the lattice and a right-hand side that is entirely perturbative:

$$\widetilde{C}_g \left(\nu, z^2 \mu^2, \mu^2 \tau \right) = \mathcal{F}^{-1}(\mu^2 \tau) \overline{C}_g \left(\nu, z^2 \mu^2 \right). \tag{15}$$

[2] I discuss the determination of $\mathcal{F}(\mu^2 \tau)$ at one loop in a separate set of notes.

0.1 A better ratio

In fact the reduced matrix element we're using is

$$\mathcal{M}(\nu, z^2) = \frac{M_g(\nu, z^2, \mu^2) M_g(0,0,\mu^2)}{M_g(0, z^2, \mu^2) M_g(P_z, 0, \mu^2)}, \tag{16}$$

which clearly fixes

$$\mathcal{M}(0, z^2) = 1. \tag{17}$$

The reduced matrix element is now

$$\begin{aligned}\mathcal{M}_g(\nu, z^2) &= \frac{\int \mathrm{d}u\, Z_g(u, z^2\mu^2) M_g(u\nu, 0, \mu^2)}{M_g(0, z^2, \mu^2)} \frac{M_g(0,0,\mu^2)}{M_g(P_z, 0, \mu^2)} + \mathcal{O}(z^2) \\ &= \frac{\int \mathrm{d}u\, Z_g(u, z^2\mu^2) M_g(u\nu, 0, \mu^2)}{\int \mathrm{d}v\, Z_g(v, z^2\mu^2) M_g(0,0,\mu^2)} \frac{M_g(0,0,\mu^2)}{M_g(P_z, 0, \mu^2)} + \mathcal{O}(z^2) \\ &= \frac{\int \mathrm{d}u\, Z_g(u, z^2\mu^2) M_g(u\nu, 0, \mu^2)}{M_g(P_z, 0, \mu^2) \int \mathrm{d}v\, Z_g(v, z^2\mu^2)} + \mathcal{O}(z^2).\end{aligned} \tag{18}$$

It seems to me that the best way to express this is then

$$\begin{aligned}M_g(P_z, 0, \mu^2)\mathcal{M}_g(\nu, z^2) &= \frac{\int \mathrm{d}u\, Z_g(u, z^2\mu^2) M_g(u\nu, 0, \mu^2)}{\int \mathrm{d}v\, Z_g(v, z^2\mu^2)} + \mathcal{O}(z^2) \\ &= C_g(z^2\mu^2) \int \mathrm{d}u\, Z_g(u, z^2\mu^2) M_g(u\nu, 0, \mu^2) + \mathcal{O}(z^2) \\ &= \int \mathrm{d}u\, \overline{Z}_g(u, z^2\mu^2) M_g(u\nu, 0, \mu^2) + \mathcal{O}(z^2) \\ &= \int \frac{\mathrm{d}u}{|\nu|} \overline{Z}_g\left(\frac{u}{\nu}, z^2\mu^2\right) M_g(u, 0, \mu^2) + \mathcal{O}(z^2).\end{aligned} \tag{19}$$

The relation between the gluon PDF and the matrix element is

$$f_{g/H}(x, \mu^2) = \int \frac{\mathrm{d}\nu}{\pi x} e^{-ix\nu} M_g(\nu, 0, \mu^2) + \mathcal{O}(z^2), \tag{20}$$

so

$$\begin{aligned}M_g(P_z, 0, \mu^2)\mathcal{M}_g(\nu, z^2, \mu^2) &= \int \frac{\mathrm{d}u}{|\nu|} \overline{Z}_g\left(\frac{u}{\nu}, z^2\mu^2\right) 2 \int \mathrm{d}x\, e^{ixu} x f_{g/H}(x, \mu^2) + \mathcal{O}(z^2) \\ &= 2 \int \mathrm{d}x \int \frac{\mathrm{d}u}{|\nu|}\, e^{ixu}\, \overline{Z}_g\left(\frac{u}{\nu}, z^2\mu^2\right) x f_{g/H}(x, \mu^2) + \mathcal{O}(z^2) \\ &= \int \frac{\mathrm{d}x}{x} 2 \int \mathrm{d}v\, e^{iv\nu}\, \overline{Z}_g\left(\frac{v}{x}, z^2\mu^2\right) x f_{g/H}(x, \mu^2) + \mathcal{O}(z^2) \\ &= \int \frac{\mathrm{d}x}{x}\, \overline{C}_g\left(\frac{\nu}{x}, z^2\mu^2\right) x f_{g/H}(x, \mu^2) + \mathcal{O}(z^2).\end{aligned} \tag{21}$$

As before, we need to relate the local matrix element in the $\overline{\text{MS}}$ scheme to that determined via the gradient flow

$$M_g(P_z, 0, \tau)\mathcal{M}_g(\nu, z^2) = \int \frac{\mathrm{d}x}{x}\, \widetilde{C}_g\left(\frac{\nu}{x}, z^2\mu^2, \mu^2\tau\right) x f_{g/H}(x, \mu^2) + \mathcal{O}(z^2), \tag{22}$$

with a left-hand side that is determined on the lattice and a right-hand side that is entirely perturbative:

$$\widetilde{C}_g\left(\nu, z^2\mu^2, \mu^2\tau\right) = \mathcal{F}^{-1}(\mu^2\tau)\overline{C}_g\left(\nu, z^2\mu^2\right). \tag{23}$$

Note we expect that $\mathcal{F}(\mu^2\tau)$ to be independent of P_z, since it is calculable in perturbation theory.

Thus I think one expects that the normalisation is, in fact, the momentum fraction plus nonperturbative lattice artefacts (although I'm not sure of the functional form)

$$\begin{aligned} M_g(P_z, 0, \tau) &= M_g(0, 0, \tau) + \mathcal{O}(a^2P_z^2) \\ &= \mathcal{F}^{-1}(\mu^2\tau)M_g(0, 0, \mu^2) + \mathcal{O}(a^2P_z^2) \\ &= \mathcal{F}^{-1}(\mu^2\tau)\langle x\rangle_g(\mu^2) + \mathcal{O}(a^2P_z^2). \end{aligned} \tag{24}$$

If this is indeed true, then the matching can probably use the published results for the energy momentum tensor at finite flow time. **However, I am not yet convinced that this is the correct approach - it still seems to me that we should treat the normalisation as $M_g(P_z, 0, \tau)$ and not $\langle x\rangle_g$.**