Extraction of PDFs with one-loop matching and systematic effects

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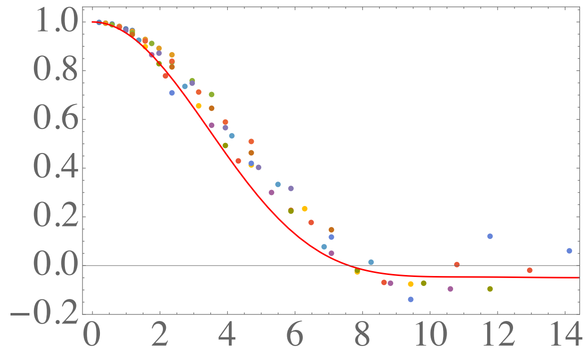
We discuss extraction of PDF with one-loop matching and systematic effects modeling

I. DATA AND PSEUDO-PDF

Data (without error bars) for the ratio $M(z,P)/M(z,0)$ are shown below for 6 momenta $P = 1, 2, 3, 4, 5.6$. The data with $z_3 > 12a$ have been excluded.



I will use in my fits normalized functions

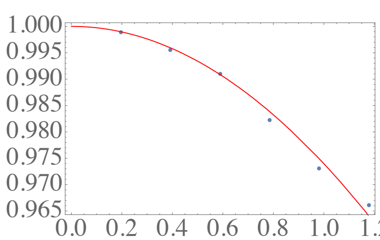
$$P(x,\alpha,\beta) = \frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+\alpha)\Gamma(1+\beta)} x^{\alpha}(1-x)^{\beta}$$

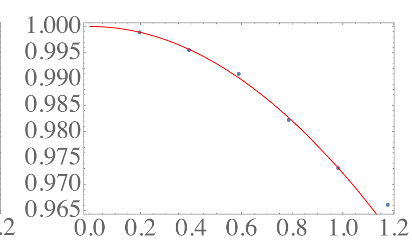
and their cosine transforms

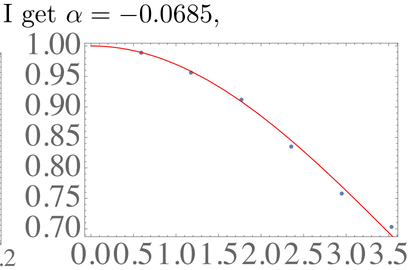
$$\mathfrak{P}(\nu;\alpha,\beta) = \int_0^1 dx\, \cos(\nu x)\, P(x,\alpha,\beta) = {}_2F_3\left(\frac{\alpha}{2}+\frac{1}{2}, \frac{\alpha}{2}+1; \frac{1}{2}, \frac{\alpha}{2}+\frac{\beta}{2}+1, \frac{\alpha}{2}+\frac{\beta}{2}+\frac{3}{2}; -\frac{\nu^2}{4}\right)$$

The curve corresponds to $\mathfrak{P}(\nu;\alpha,\beta)$ for $\alpha = 0.35$, $\beta = 3$.
To get a feeling about data at different z, I fit the data for fixed z using the functions $\mathfrak{P}(\nu;\alpha,\beta=3)$.

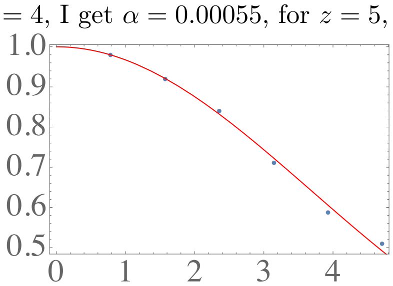
For $z = 1$, I get $\alpha = -0.1836$, for $z = 2$, I get $\alpha = -0.13226$, for $z = 3$, I get $\alpha = -0.0685$,

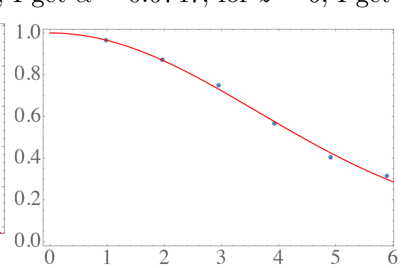


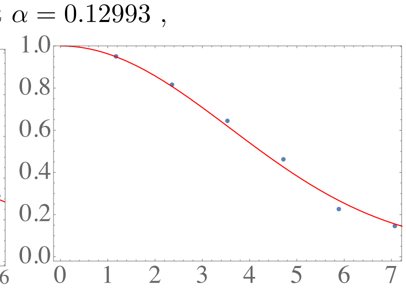




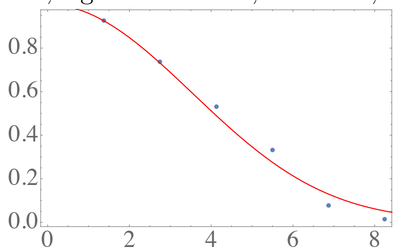
for $z = 4$, I get $\alpha = 0.00055$, for $z = 5$, I get $\alpha = 0.0747$, for $z = 6$, I get $\alpha = 0.12993$,

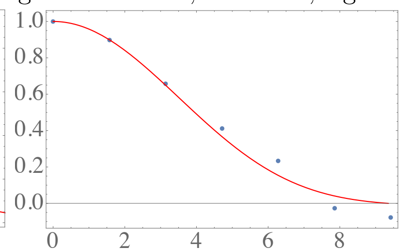


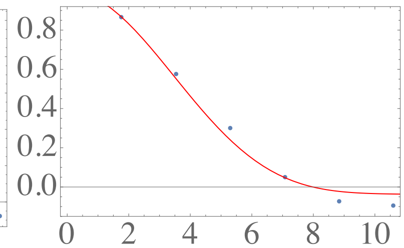




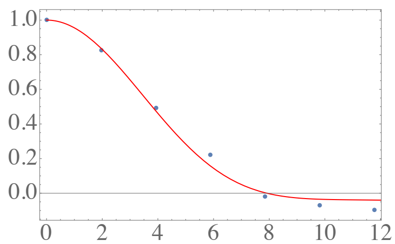
for $z = 7$, I get $\alpha = 0.1995$, for $z = 8$, I get $\alpha = 0.2500$, for $z = 9$, I get $\alpha = 0.31785$.

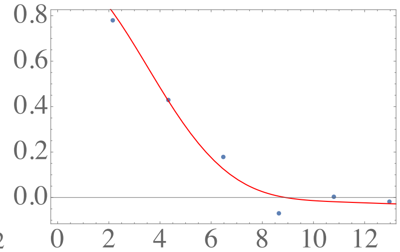


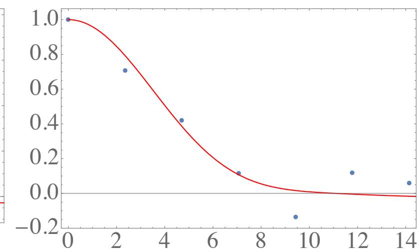




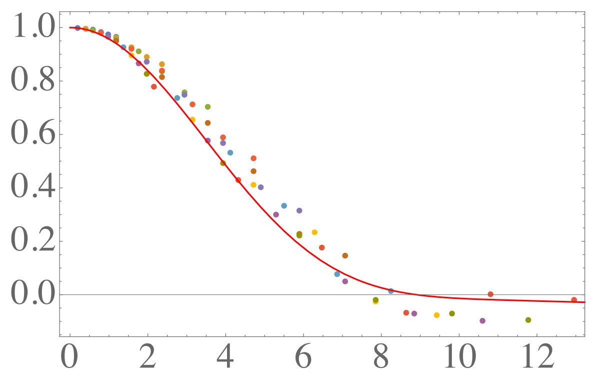
for $z = 10$, I get $\alpha = 0.3230$, for $z = 11$, I get $\alpha = 0.2661$, for $z = 12$, I get $\alpha = 0.2120$.



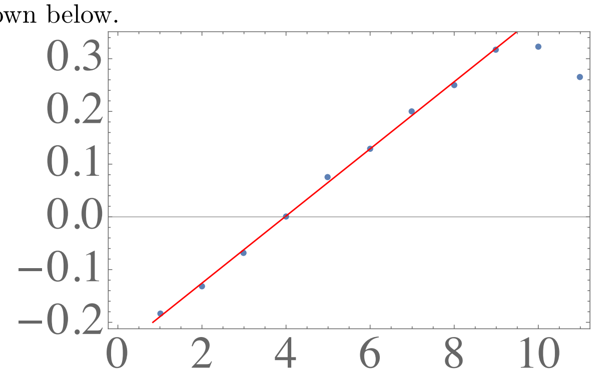




In all cases, fits go reasonably close to data at that particular z. However, if the data with smaller z are also shown, the picture is not so impressive.



The change of α with z is shown below.



Till $z = 9$, it is well described by a straight line

$$\alpha(z) = -0.2530 + 0.06366z \approx 0.06366(z - 4)$$

Which means that the data (for positive z_3) are well described by the pseudo-PDF

$$\mathcal{P}(x, z_3^2) = \frac{\Gamma(4 + \alpha(z_3))}{3!\Gamma(1 + \alpha(z_3))} x^{\alpha(z_3)}(1 - x)^3$$

Naively, we can say that PDF $f(x)$ is given by $z_3 = 0$ extrapolation of this function, i.e., by

$$f(x) = \frac{\Gamma(4 - 0.253)}{3!\Gamma(1 - 0.253)} x^{-0.253}(1 - x)^3$$

We notice that the power $\alpha(z)$ decreases with z, as it should if we trust the AP evolution for pseudo-PDF. So, let us see if this pattern of z-dependence is numerically compatible with the AP evolution equation. In any case, we need to convert pseudo-PDF into MS-bar PDF, thus let us discuss the conversion from the pseudo-ITD to MS-bar ITD.

II. CHECKING EVOLUTION

Basic factorization formula given by OPE:

$$\mathfrak{M}^{\rm data}(\nu, z_3^2) = \mathcal{I}(\nu,\mu^2) - \frac{\alpha_s}{2\pi}\, C_F \int_0^1 du\, \mathcal{I}(u\nu, \mu^2) T(u, z_3^2\mu^2) \ ,$$

To get "data points" for the MS-bar ITD, one should use

$$\mathcal{I}(\nu,\mu^2) = \mathfrak{M}^{\rm data}(\nu, z_3^2) + \frac{\alpha_s}{2\pi}\, C_F \int_0^1 du\, \mathcal{I}(u\nu, \mu^2) T(u, z_3^2\mu^2) \ ,$$

Since $\mathcal{I}(u\nu, \mu^2)$ is also present on the rhs, we may proceed by iterations

$$\mathcal{I}(\nu,\mu^2) = \mathfrak{M}^{\rm data}(\nu, z_3^2) + \frac{\alpha_s}{2\pi}\, C_F \int_0^1 du\, \mathcal{I}_0(u\nu) T(u, z_3^2\mu^2) \ ,$$

where $\mathcal{I}_0(\nu)$ is a (rough) fit of data for $\mathfrak{M}^{\rm data}(\nu, z_3^2)$ by a combination of functions $\mathfrak{P}(\nu;\alpha,\beta)$ and

$$T(u, z_3^2\mu^2) = \left[\frac{1+u^2}{1-u}\right]_+ \ln\left(z_3^2\mu^2\frac{e^{2\gamma_E+1}}{4}\right) + \left[\frac{4\ln(1-u)}{1-u} - 2(1-u)\right]_+$$

As we have seen, $\mathfrak{M}^{\rm data}(\nu, z_3^2)$ is given by $\mathfrak{P}(\nu;\alpha(z_3),\beta)$ with $\alpha(z_3) = -0.2530 + 0.06366 z_3$ and $\beta = 3$. For $-1 < \alpha < 1$, $\mathfrak{P}(\nu;\alpha,3)$ is a linear function of α with a good precision.

One can also check that, for small α we have the following relation

$$\int_0^1 du\, \mathfrak{P}(u\nu;\alpha,3)\left[\frac{1+u^2}{1-u}\right]_+ \approx -2.1\frac{\partial}{\partial\alpha}\mathfrak{P}(\nu;\alpha,3) \equiv -2.1\mathfrak{p}(\nu)$$

Hence, we have

$$\begin{aligned}
\mathcal{I}(\nu,\mu^2) &\approx \mathfrak{P}(\nu;0,3) + \alpha(z_3)\mathfrak{p}(\nu) - 2.1\frac{\alpha_s}{2\pi}\, C_F \ln\left(z_3^2\right)\mathfrak{p}(\nu) + \mathfrak{f}(\nu,\mu) \\
&= \mathfrak{P}(\nu;0,3) + (\alpha(z_3) - 2.1 C_F\frac{\alpha_s}{\pi}\ln z_3)\mathfrak{p}(\nu) + \mathfrak{f}(\nu,\mu) \\
&\approx \mathfrak{P}(\nu;\alpha_{\rm ev}(z_3),3) \ ,
\end{aligned}$$

where

$$\alpha_{\rm ev}(z_3) \approx \alpha(z_3) - 2.1 C_F\frac{\alpha_s}{\pi}\ln z_3 + \mathfrak{f}(\nu)/\mathfrak{p}(\nu,\mu) \ ,$$

determines the z_3-dependence of the "matched" ITD. In a more explicit form

$$\alpha_{\rm ev}(z_3) \approx -0.2530 + 0.06366 z_3 - 3\frac{\alpha_s}{\pi}\ln z_3 + \mathfrak{f}(\nu,\mu)/\mathfrak{p}(\nu) \ ,$$

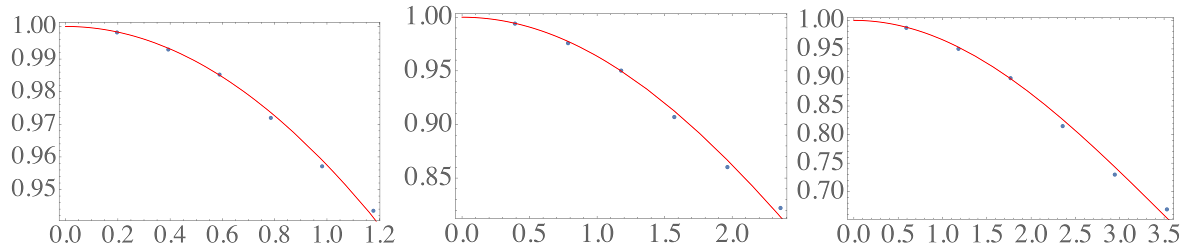
and it is clear that the "matched" ITD has z_3-dependence, unfortunately. Still, the function $\alpha_{\rm ev}(z_3)$ has a minimum for $z_3 \sim 4$, and may be treated as a constant within ± 0.05 for $2 \le z_3 \le 8$.

III. CONVERTING TO PDF

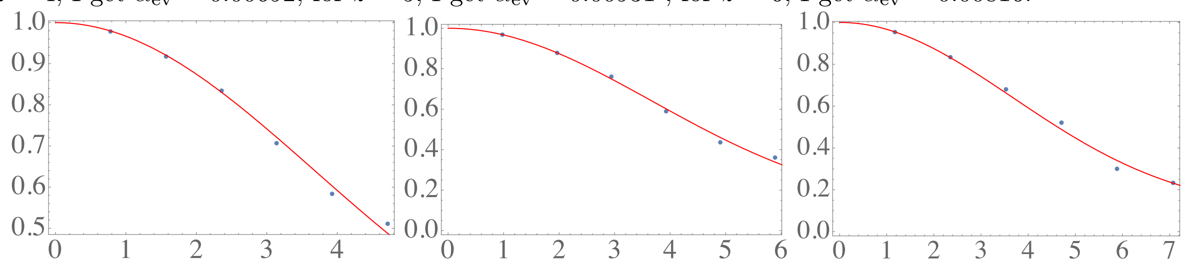
Let us check these expectations by an explicit conversion to MS-bar ITD.

I will use $\mathcal{I}_0(\nu) = \mathfrak{P}(\nu; 0.2, 3)$, $\alpha_s/\pi = 0.1$ and $\mu = 1/a \approx 2$ GeV to get points for $\mathcal{I}(\nu, \mu^2)$. The results are shown below for z up to 12.

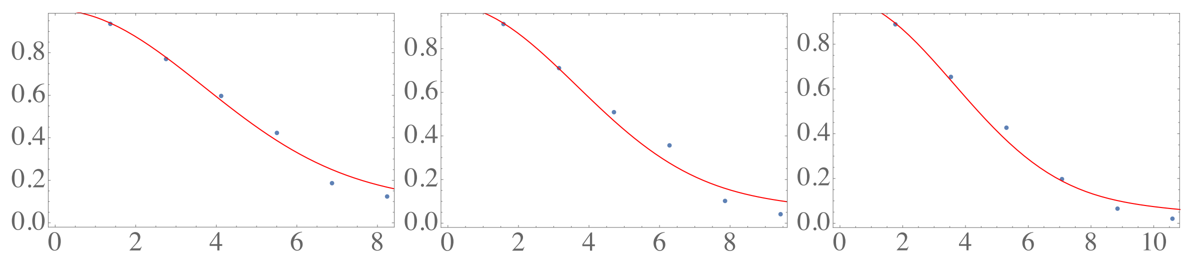
Fitting the evolved data for each fixed z using $\mathfrak{P}(\nu; \alpha, 3)$, I get the following z-dependence in $\alpha_{\rm ev}$. For $z = 1$, I get $\alpha_{\rm ev} = 0.2623$, for $z = 2$, I get $\alpha = 0.0981$, for $z = 3$, I get $\alpha = 0.03135$,



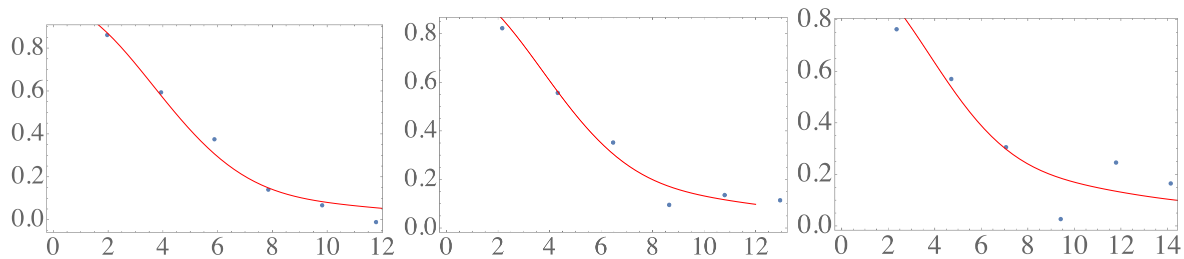
For $z = 4$, I get $\alpha_{\rm ev} = 0.00692$, for $z = 5$, I get $\alpha_{\rm ev} = 0.00931$, for $z = 6$, I get $\alpha_{\rm ev} = 0.00816$.



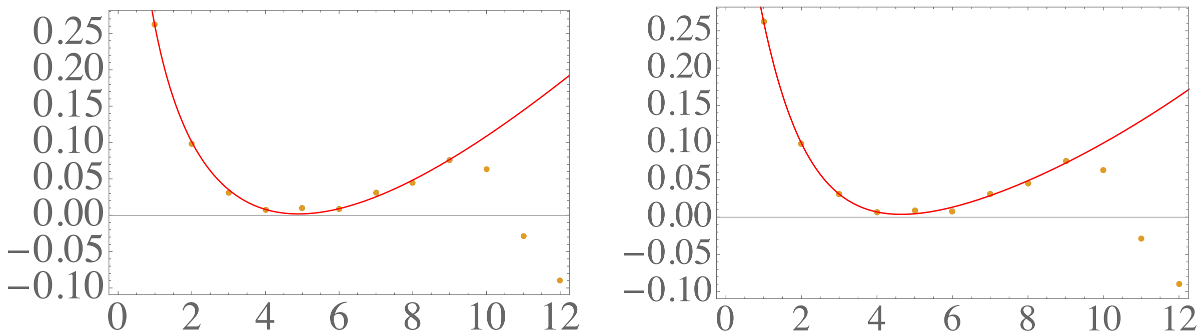
As we can see, the powers $\alpha_{\rm ev}$ are rather close to each other, with the exception of $z = 1$ case. Continuing to higher z, for $z = 7$, I get $\alpha_{\rm ev} = 0.03146$, for $z = 8$, I get $\alpha_{\rm ev} = 0.04525$, for $z = 9$, I get $\alpha_{\rm ev} = 0.0754$.



For $z = 10$, I get $\alpha_{\rm ev} = 0.06327$, for $z = 11$, I get $\alpha_{\rm ev} = -0.02827$, for $z = 12$, I get $\alpha_{\rm ev} = -0.08977$.



Now plot for $\alpha_{\rm ev}(z)$



The curve shown on the left is a logarithmic fit for points with $z \leq 9$

$$\alpha_{\rm ev}(z) = 0.1934 + 0.0666z - 0.3256 \ln z \ ,$$

which is in full agreement with the expression for $\alpha_{\rm ev}(z)$ derived in the previous section. The points above $z = 9$ start to deviate from this pattern, but still are within ± 0.05 from the minimum of $\alpha_{\rm ev}(z)$. Also, since large z's are involved, these points will have large error bars.

IV. SUMMARY

Thus, the original data show a very simple, practically linear dependence on z_3 of the effective power $\alpha(z_3)$. Performing matching, we add $\ln 1/z_3$ terms to it, resulting in a function $\alpha_{\rm ev}(z_3)$ having a minimum around $z_3 = 5a$. This produces an approximately constant ± 0.05 behavior of $\alpha_{\rm ev}(z_3)$ in the region $2 \leq z_3 \leq 8$. This looks like an approximate compliance with the AP evolution equation. However, for $z_3 = 1$, we have ~ 0.25 deviation of $\alpha_{\rm ev}(z_3)$ from its minimum. Explanation is that even if $\ln z_3$ can imitate a close to linear behavior for $z_3 > 2a$, it cannot do it everywhere.

The question is why the data do not show a logarithmic behavior in z_3 in the region of small z_3, where the short-distance OPE is expected to work best?

A possible answer is that for $z_3 = a$, the data may be affected by finite lattice spacing effects. In fact, we can fit $\alpha_{\rm ev}(z)$ in another way, namely, using

$$\alpha_{\rm ev}(z) = -0.043773 + 0.67523e^{-0.797z} + 0.00143z^2$$

(see the right panel) and "explain" the curve by discretization effects described by $e^{-0.797z/a}$ and by a higher-twist term $0.00143z^2$. In the latter, the scale in $0.00143\, z^2/a^2 \approx z^2\,(80\,\text{MeV})^2$ is small, so there are no large higher-twist effects visible.

The fast fall-off of $e^{-0.797z/a}$ may be imitated by inverse powers $(a/z)^n$. Hence, adding such *ad hoc* terms in the OPE, we can get a decent fit of the data.

The key lesson is that the lattice data does not show the expected $\ln z$ behavior in the effective power $\alpha(z)$ (i.e., in the original data), and all the small-z peculiarities of the matched ITD are brought in by the $\ln z$ term in the matching relation.

One may try to do a two-parameter fitting in α and β, and see what will happen in that case. I think it is unlikely that the basic observation $\mathfrak{M}^{\rm data}(\nu, z_3^2) = M_0(\nu) + z_3 M_1(\nu)$ will change.

One may also try to improve the fitting of the data for a fixed z by using Ansätze more complicated than $x^\alpha(1-x)^\beta$, which may somewhat improve χ^2. However, from what I see, the main reason for large χ^2 is fitting the original data by an expression containing $\ln z$ in a situation when the data in fact do not show such a term.

It looks like the easiest way to improve χ^2 is to throw out the $z = 1$ data from the fit.