GPD matrix element fits

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I detail below a different method to fit the bare matrix elements compared to the sum of ratios of 3-pt and 2-pt functions. It is the first time I do this kind of extraction, so there might be rookie mistakes. However, the results are different enough that they might deserve attention. For the randomly chosen momenta $\vec{p_i} = (0, 1, 0)$, $\vec{p_f} = (2, 1, 1)$ and $z_3 = 2$ used to illustrate this document, the linear fit of the ratio starting from $T_{min} = 6$ gives:

$$\langle p_f | \mathcal{O}(z_3) | p_i \rangle = 0.0777 \pm 0.0026 \,, \tag{1}$$

whereas the method proposed in this document gives:

$$\langle p_f | \mathcal{O}(z_3) | p_i \rangle = 0.0982 \pm 0.0023 \,, \tag{2}$$

that is a difference of the order of 8σ.

2-pt functions

First, I perform a multi-exponential fit of the 2-pt functions to extract the energies $E_n(\vec{p})$ and amplitudes $Z_n(\vec{p})$ of the first few states. For $\vec{p} = (0, 0, 0)$, the results are the following:

Number of excited states	Energies (lattice units)	Uncertainties	χ^2/ d.o.f. of the fit
0	0.7095	0.0505	123446
1	0.5501	0.0035	321
	2.3334	0.0639	
2	0.5336	0.0006	1.12
	1.1610	0.0248	
	3.3630	0.0802	
3	0.5243	0.1340	0.2703
	0.6513	0.0104	
	1.3559	0.0993	
	3.7711	0.2305	

A simple observation of the χ^2 demonstrates that the fit with two excited states reproduces the data satisfactorily, whereas three excited states is over-fitting and less than two is grossly underfitting. The result is shown on Fig. 1. The precision of the reconstruction of the ground state at rest with two excited states is of the per-mille level. The proton mass is 1.117 GeV.

I perform the same extraction with two excited states for the 71 values of momenta available in the 2-pt functions. Only the two-excited states fit is able to reproduce accurately the dispersion relation of the ground state, at the level of 1%. The results are shown on Fig. 2.

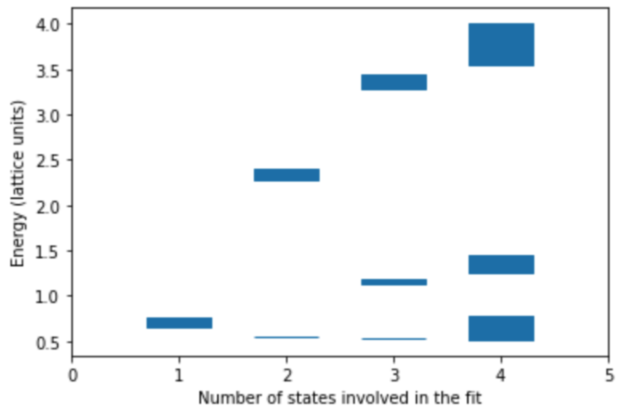


Figure 1: Energy extracted at rest for a different number of states in the fit. The width of the bands represents the uncertainty on the energy level. The first two levels in the last case are indistinguishable due to their uncertainty.

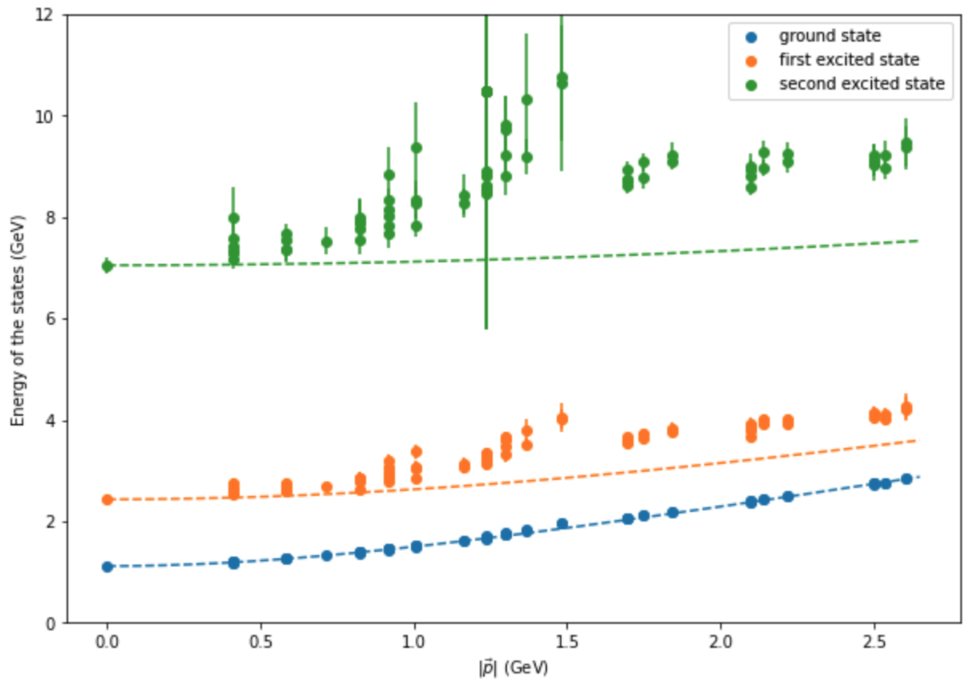


Figure 2: $E_0(\vec{p})$ (blue), $E_1(\vec{p})$ (orange) and $E_2(\vec{p})$ (green) for all momenta available. Several directions of $\vec{p}$ correspond to the same norm. The dotted line are the continuum dispersion relations, which we don't expect the excited states to follow as they are intrinsically multi-particle states.

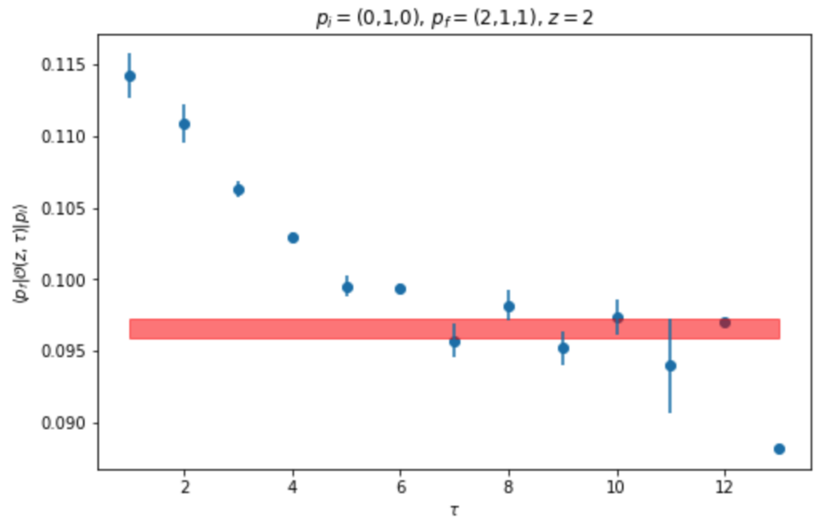


Figure 3: Ground state to ground state matrix element depending on the time-slice of the operator insertion τ, and an extraction of its large τ limit. This is a one-parameter fit.

3-pt functions

Assuming that the amplitudes are real positive numbers, this information will be used to extract the matrix element from the 3-pt function. First, I will only consider the ground-state to ground-state matrix element. Then

$$C_{3pt}(\vec{p}_i, \vec{p}_f, z_3; T, \tau) = \frac{Z_0(\vec{p}_i) Z_0(\vec{p}_f)}{4E_0(\vec{p}_i)E_0(\vec{p}_f)} \langle p_f|\mathcal{O}(z_3, \tau)|p_i\rangle e^{-(E_0(\vec{p}_i)-E_0(\vec{p}_f)\tau} e^{-E_0(\vec{p}_f)T} . \tag{3}$$

At fixed τ, this is really a linear fit of a single parameter. At first, I will neglect the uncertainty on the energies and amplitudes of the 2-pt functions, and consider their central value as an exact determination. Then, the fitted values of $\langle p_f|\mathcal{O}(z_3, \tau)|p_i\rangle$ are shown on Fig. 3.

The final result is given by the limit of large τ of $\langle p_f|\mathcal{O}(z_3, \tau)|p_i\rangle$. However, when τ is large, there is only a small number of data points $T > \tau$ to perform the fit. For this reason, I discard the last two points (where the fit of the single parameter is actually performed on a single data point). Instead, I fit a constant on the matrix elements extracted for $\tau \in \{7, 11\}$ and obtain the red band:

$$\langle p_f|\mathcal{O}(z_3)|p_i\rangle = 0.0965 \pm 0.0006 . \tag{4}$$

However, this extraction ignores the uncertainty of the energies and amplitudes extracted from the 2-pt functions, and does not make any use of the excited states. The ability to take into account higher state contamination is in fact quite limited. Extracting the τ-dependent matrix element at fixed τ from Eq. (3) is performed solely thanks to the T-dependence of the 3-pt function. However, the T dependence is exactly the same if the final state is the same. Therefore, one cannot disentangle any of the "excited state to ground state" contributions. Only an hypothesis on the τ dependence of the matrix element could allow to go forward in that direction. Doesn't the summation method make an implicit assumption on that? Is it well grounded?

On the other hand, it is possible to include contributions "ground state to excited state", since they have a different T behavior. Fig. 4 shows the result on the ground state to ground state matrix element of the inclusion of this first excited state contamination. It is now a two-parameter fit, so

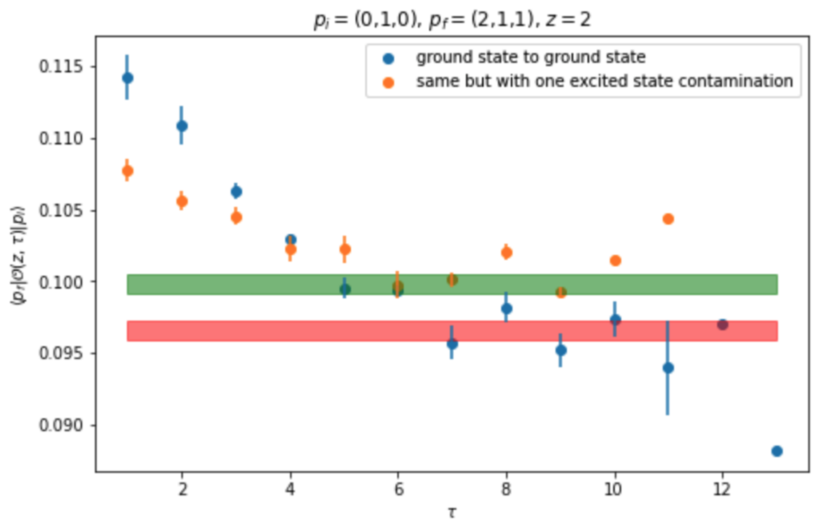


Figure 4: Same caption as Fig. 3, except I added a free parameter for the orange plot: the ground state to first excited state matrix element. The large τ limit of the new extraction of the ground state to ground state matrix element is depicted by the green band.

the last two values of τ cannot be fitted. The large τ behavior is obtained by fitting a constant on the data at $\tau \in \{7, 9\}$. The new result is:

$$\langle p_f|\mathcal{O}(z_3)|p_i\rangle = 0.0998 \pm 0.0007\,. \tag{5}$$

Out of a better idea, I will give an uncertainty to the final result such that the 1σ band exactly includes the two bands presented in Fig. 4, that is

$$\langle p_f|\mathcal{O}(z_3)|p_i\rangle = 0.0982 \pm 0.0023\,. \tag{6}$$

Adding the "ground state to second excited state" in the fit makes no difference as observed on Fig. 5. The uncertainty on the well-determined 2-pt function extraction should represent a fairly minor modification compared to the uncertainty I have added here. 108 fits for $z = 0$ and plenty of configurations of momentum are shown in the final pages following the same codes as Fig. 4.

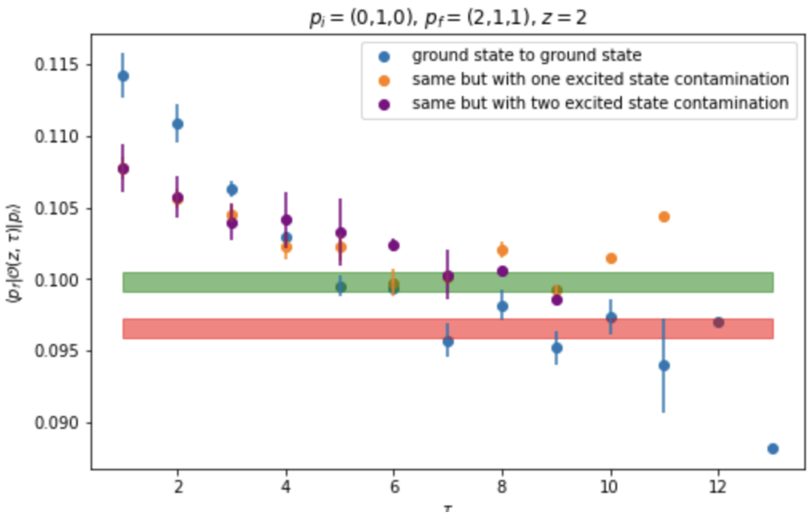
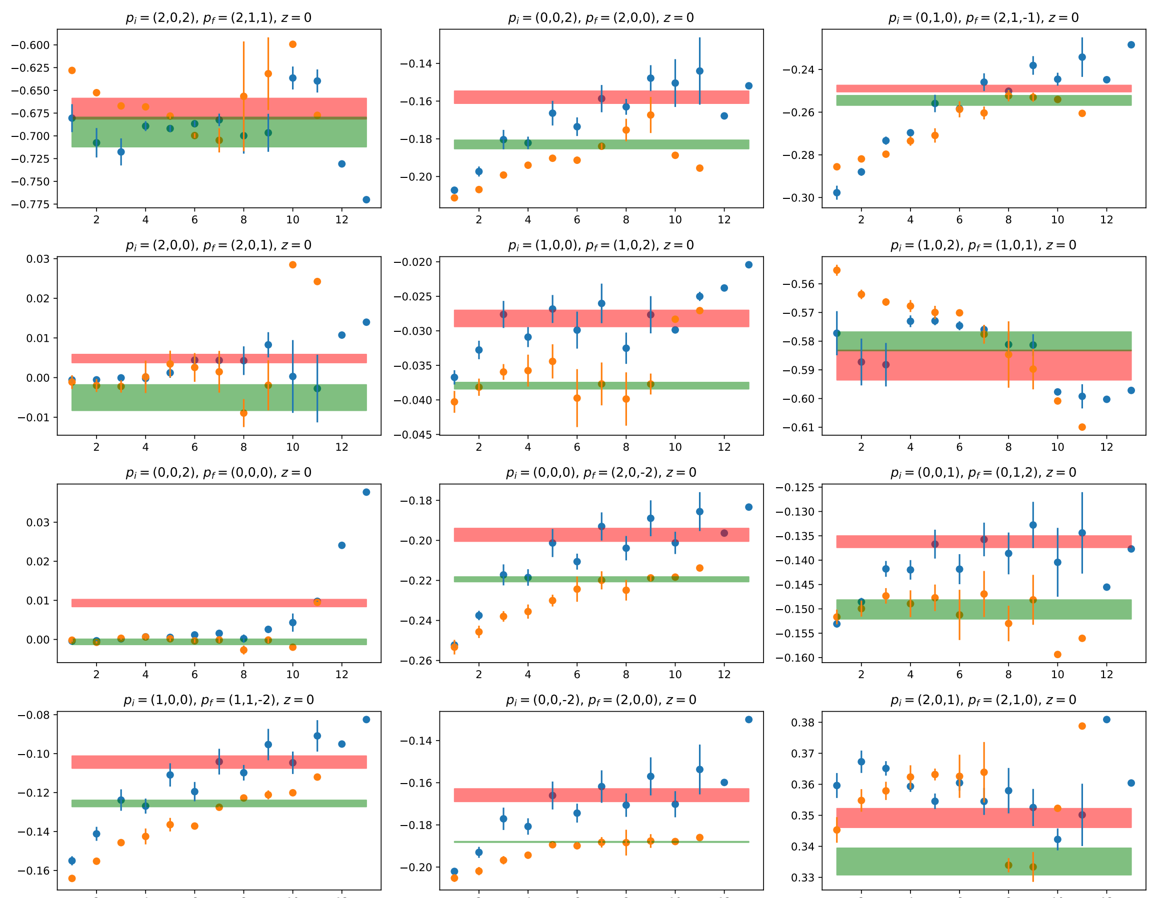
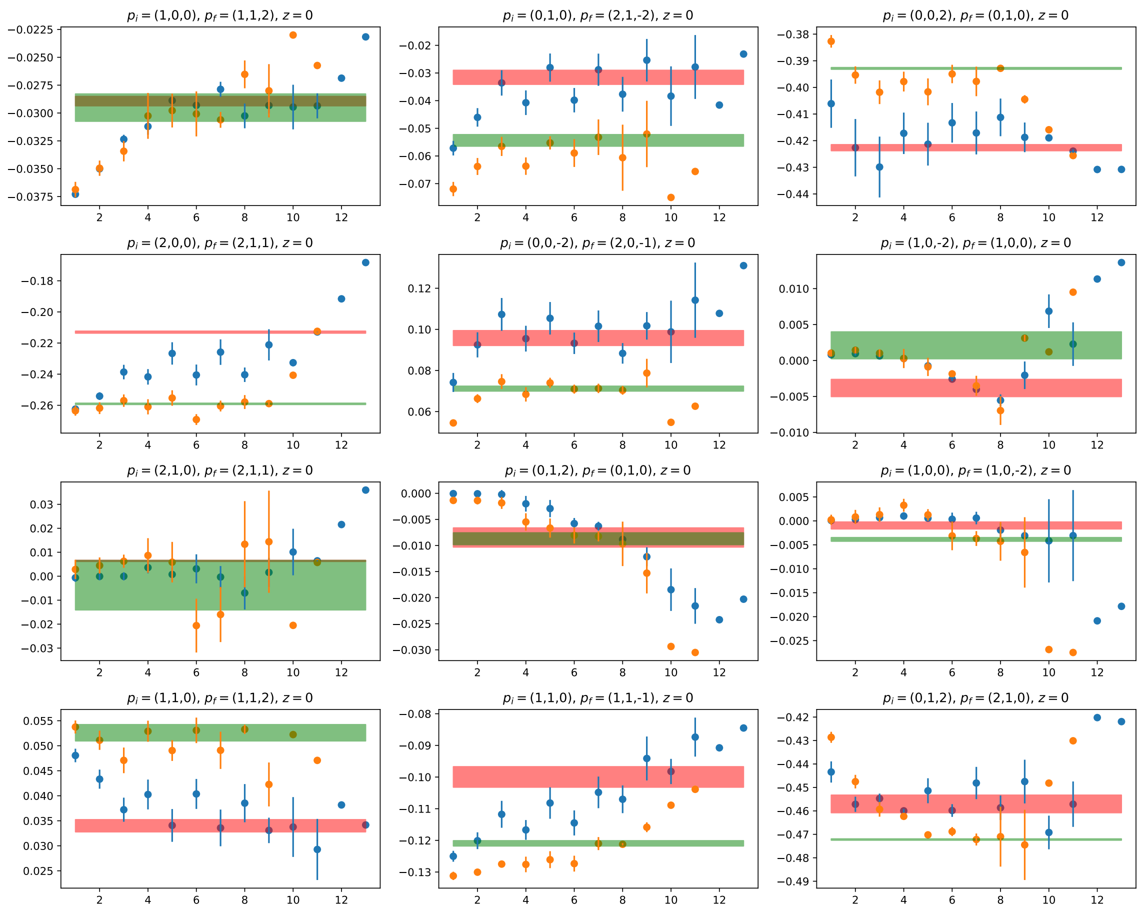


Figure 5: The inclusion of the ground state to second excited state matrix element barely modifies the fit (purple points).

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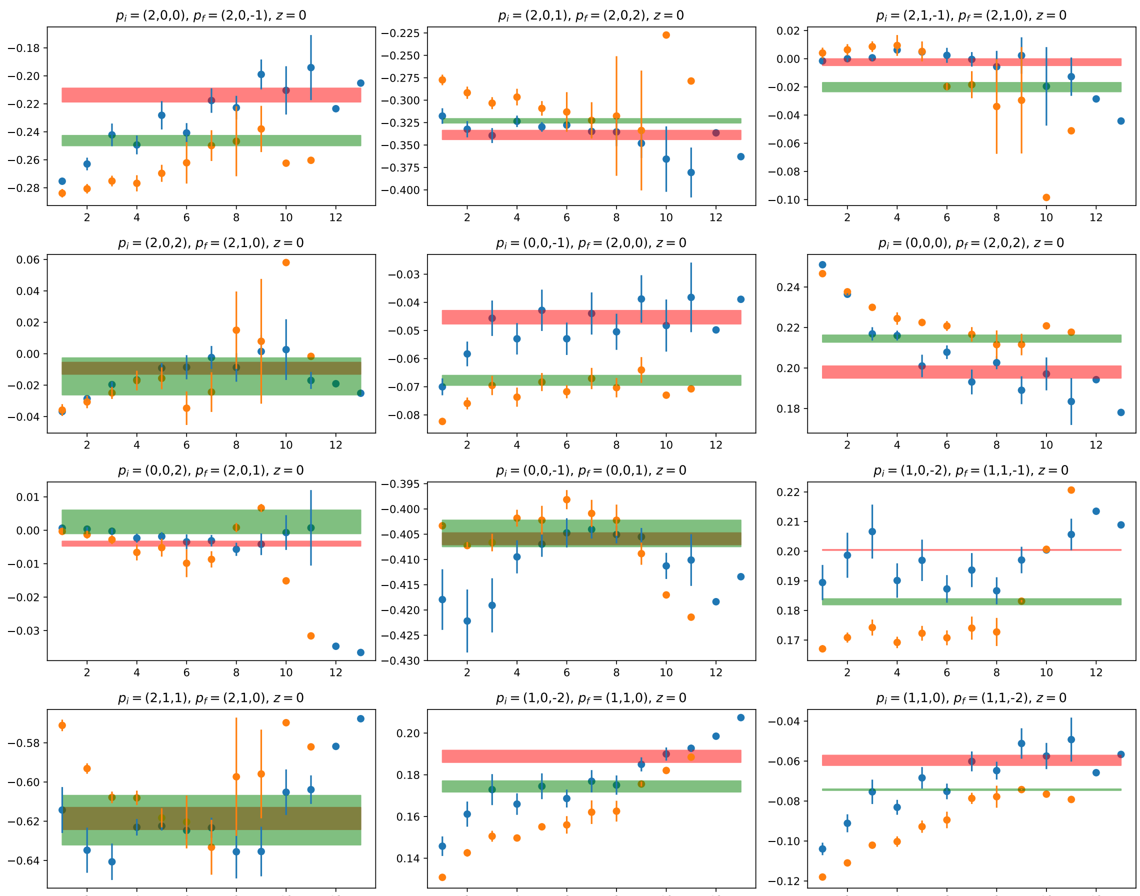

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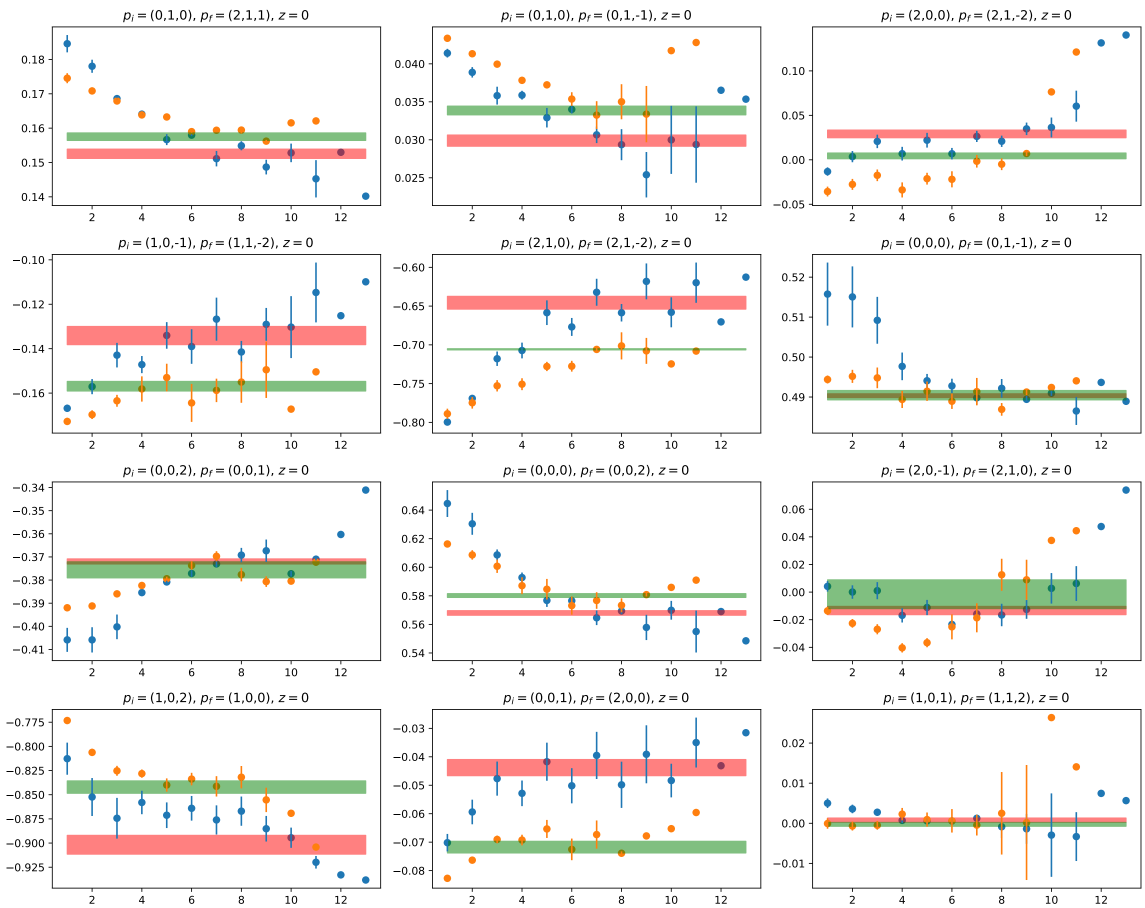

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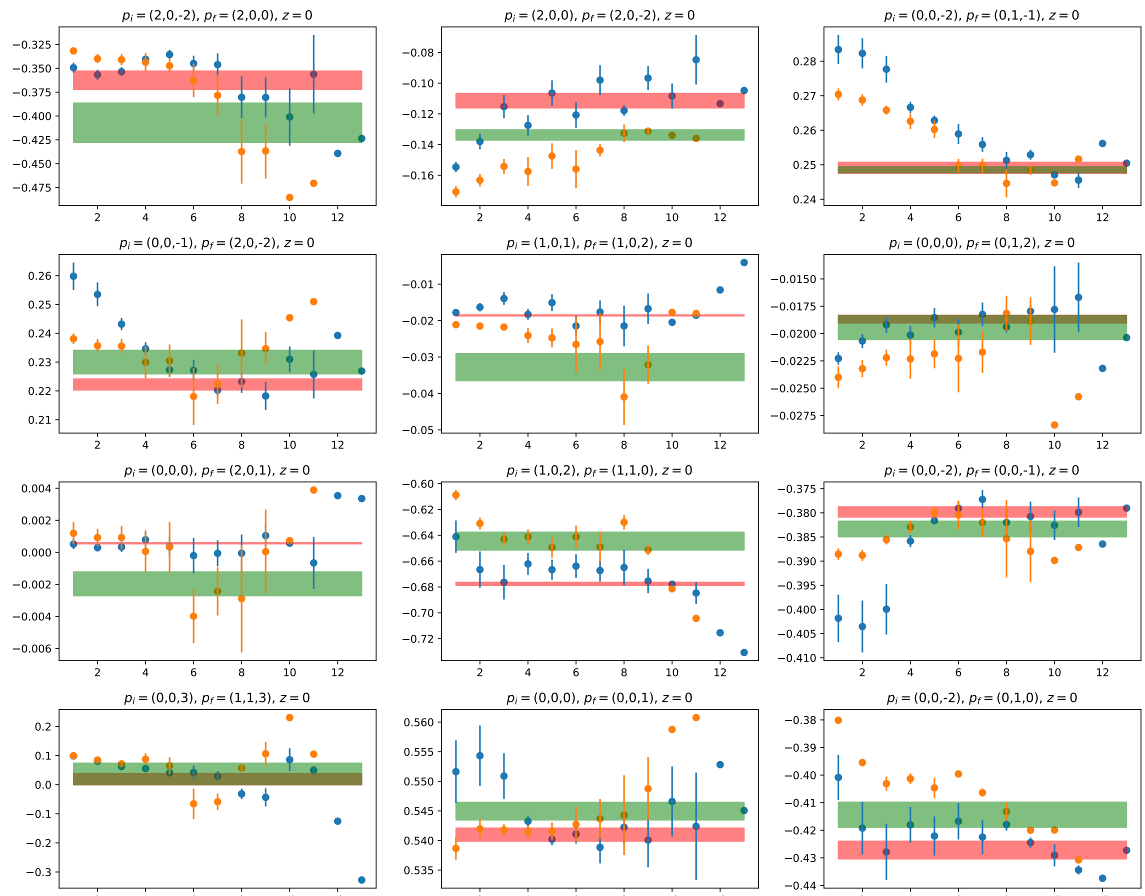
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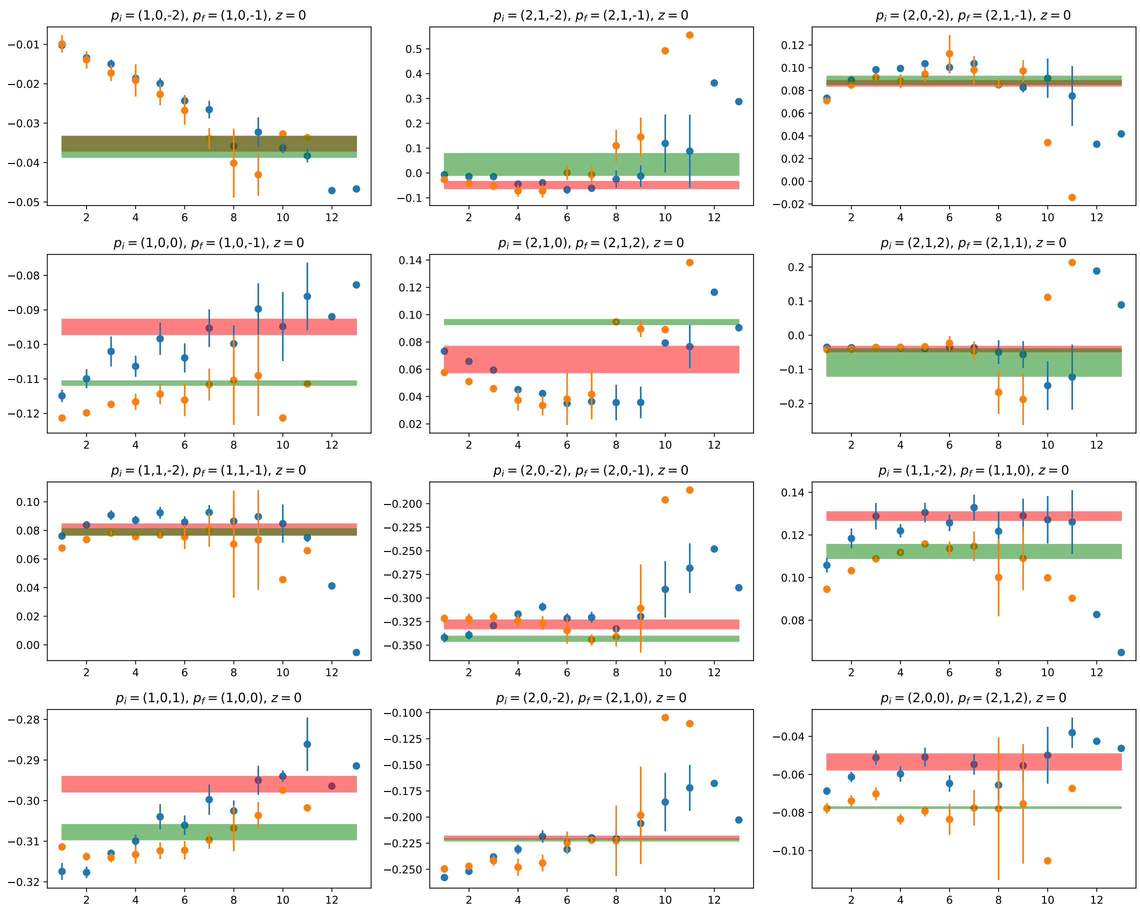
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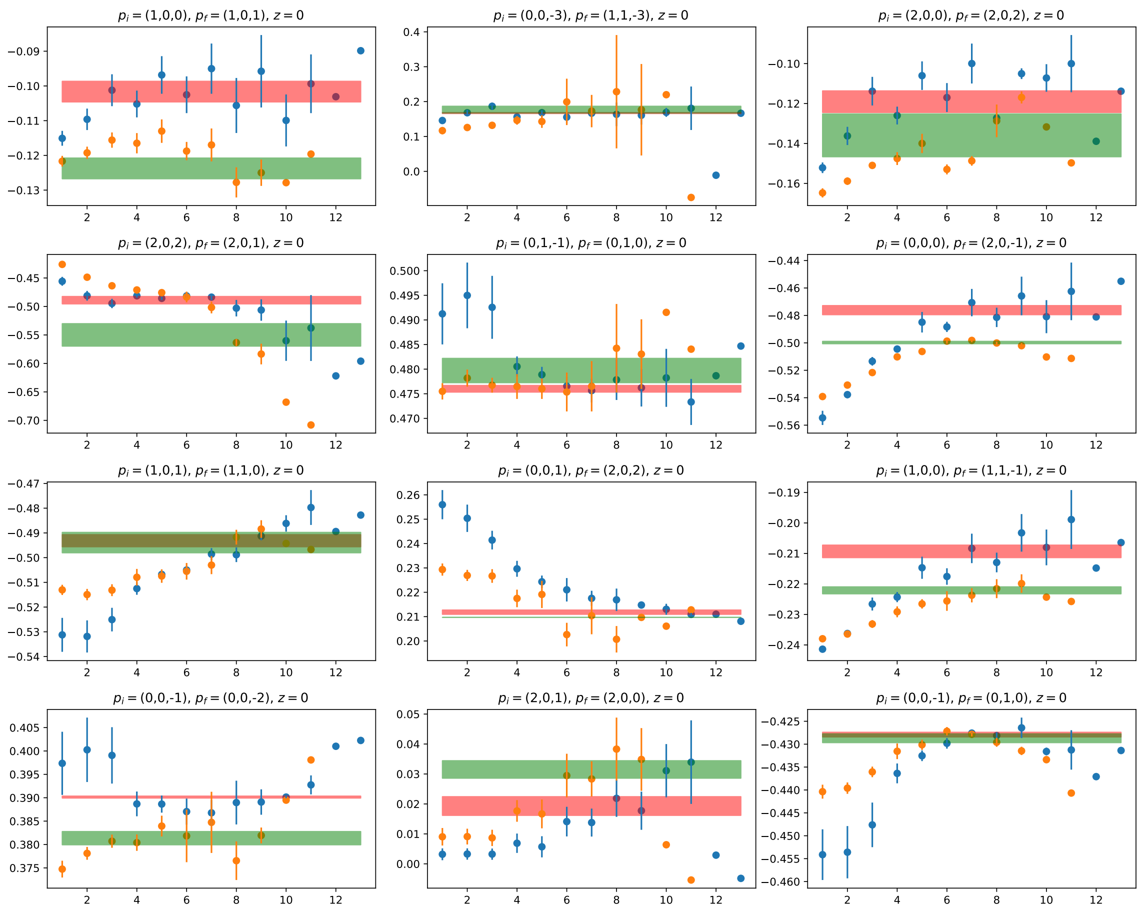
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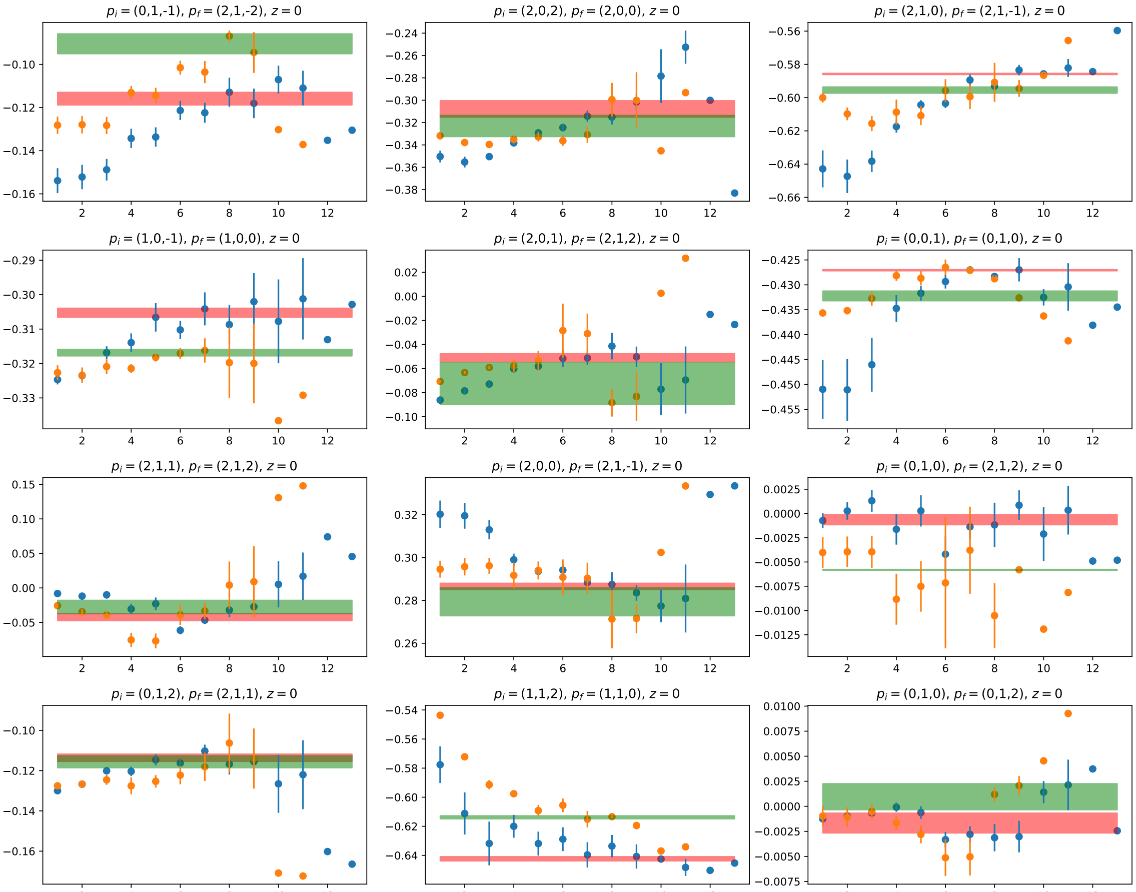

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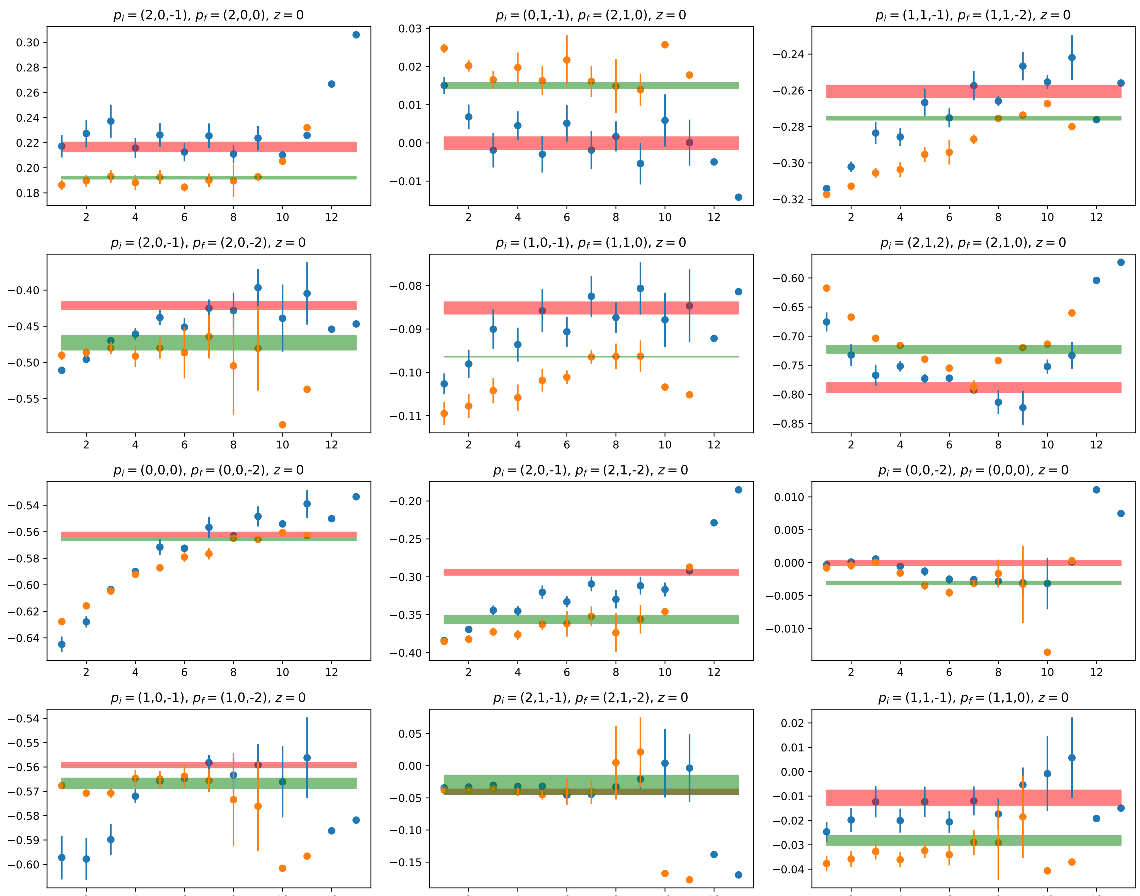
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