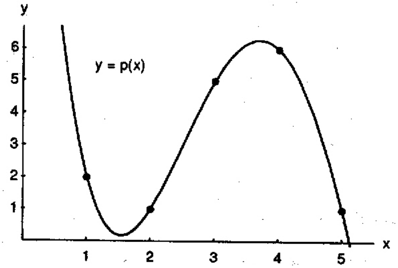
Interpolation, Extrapolation & Polynomial Approximation

November 10, 2014

Introduction

In many cases we know the values of a function $f(x)$ at a set of points $x_1, x_2, \ldots, x_N$, but we don't have the analytic expression of the function that lets us calculate its value at an arbitrary point. We will try to estimate $f(x)$ for arbitrary x by "drawing" a curve through the x_i and sometimes beyond them.



The procedure of estimating the value of $f(x)$ for $x \in [x_1, x_N]$ is called interpolation while if the value is for points $x \notin [x_1, x_N]$ extrapolation.

The form of the function that approximates the set of points should be a convenient one and should be applicable to a general class of problems.

Polynomial Approximations

Polynomial functions are the most common ones while rational and trigonometric functions are used quite frequently.

We will study the following methods for polynomial approximations:

- Lagrange's Polynomial
- Hermite Polynomial
- Taylor Polynomial
- Cubic Splines

Lagrange Polynomial

Let's assume the following set of data:

	x_0	x_1	x_2	x_3
x	3.2	2.7	1.0	4.8
$f(x)$	22.0	17.8	14.2	38.3
	f_0	f_1	f_2	f_3

Then the interpolating polynomial will be of 4th order i.e. $ax^3 + bx^2 + cx + d = P(x)$. This leads to 4 equations for the 4 unknown coefficients and by solving this system we get $a = -0.5275$, $b = 6.4952$, $c = -16.117$ and $d = 24.3499$ and the polynomial is:

$$P(x) = -0.5275x^3 + 6.4952x^2 - 16.117x + 24.3499$$

It is obvious that this procedure is quite laboureus and Lagrange developed a direct way to find the polynomial

$$P_n(x) = f_0 L_0(x) + f_1 L_1(x) + ... + f_n L_n(x) = \sum_{i=0}^{n} f_i L_i(x) \qquad (1)$$

where $L_i(x)$ are the **Lagrange coefficient polynomials**

$$L_j(x) = \frac{(x-x_0)(x-x_1)...(x-x_{j-1})(x-x_{j+1})...(x-x_n)}{(x_j-x_0)(x_j-x_1)...(x_j-x_{j-1})(x_j-x_{j+1})...(x_j-x_n)} \quad (2)$$

and obviously:

$$L_j(x_k) = \delta_{jk} = \begin{cases} 0 & \text{if} \quad j \neq k \\ 1 & \text{if} \quad j = k \end{cases}$$

where δ_{jk} is Kronecker's symbol.

ERRORS

The error when the Lagrange polynomial is used to approximate a continuous function $f(x)$ it is :

$$E(x) = (x-x_0)(x-x_1)....(x-x_n)\frac{f^{(n+1)}(\xi)}{(n+1)!} \quad \text{where} \quad \xi \in [x_0, x_N] \quad (3)$$

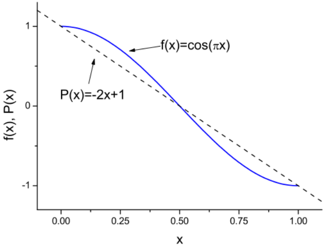
NOTE

Lagrange polynomial applies for evenly and unevenly spaced points. Still if the points are evenly spaced then it reduces to a much simpler form.

Lagrange Polynomial : Example

Find the Lagrange polynomial that approximates the function $y = \cos(\pi x)$.
We create the table

x_i	0	0.5	1
f_i	1	0.0	-1



The Lagrange coeffiecient polynomials are:

$$L_1(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} = \frac{(x-0.5)(x-1)}{(0-0.5)(0-1)} = 2x^2 - 3x + 1,$$

$$L_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} = \frac{(x-0)(x-1)}{(0.5-0)(0.5-1)} = -4(x^2 - x)$$

$$L_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} = \frac{(x-0)(x-0.5)}{(1-0)(1-0.5)} = 2x^2 - x$$

thus

$$P(x) = 1 \cdot (2x^2 - 3x + 1) - 0.4 \cdot (x^2 - x) + (-1) \cdot (2x^2 - x) = -2x + 1$$

The error will be:

$$E(x) = x \cdot (x - 0.5) \cdot (x - 1) \frac{\pi^3 \sin(\pi\xi)}{3!}$$

e.g. for $x = 0.25$ is $E(0.25) \leq 0.24$.

Forward Newton-Gregory

$$\begin{aligned} P_n(x_s) &= f_0 + s \cdot \Delta f_0 + \frac{s(s-1)}{2!} \cdot \Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!} \cdot \Delta^3 f_0 + \ldots \\ &= f_0 + \binom{s}{1} \cdot \Delta f_0 + \binom{s}{2} \cdot \Delta^2 f_0 + \binom{s}{3} \cdot \Delta^3 f_0 + \ldots \\ &= \sum_{i=0}^{n} \binom{s}{i} \cdot \Delta^i f_0 \qquad (4) \end{aligned}$$

where $x_s = x_0 + s \cdot h$ and

$$\Delta f_i = f_{i+1} - f_i \qquad (5)$$

$$\Delta^2 f_i = f_{i+2} - 2f_{i+1} + f_i \qquad (6)$$

$$\Delta^3 f_i = f_{i+3} - 3f_{i+2} + 3f_{i+1} - f_i \qquad (7)$$

$$\Delta^n f_i = f_{i+n} - nf_{i+n-1} + \frac{n(n-1)}{2!} f_{i+n-2} - \frac{n(n-1)(n-2)}{3!} f_{i+n-3} + \cdot \qquad (8)$$

ERROR The error is the same with the Lagrange polynomial :

$$E(x) = (x - x_0)(x - x_1)\dots(x - x_n)\frac{f^{(n+1)}(\xi)}{(n+1)!} \quad \text{where} \quad \xi \in [x_0, x_N] \quad (9)$$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0.0	0.000					
		0.203				
0.2	0.203		0.017			
		0.220		0.024		
0.4	0.423		0.041		0.020	
		0.261		0.044		0.032
0.6	0.684		0.085		0.052	
		0.246		0.096		
0.8	1.030		0.181			
		0.527				
1.0	1.557					

Table: Example of a difference matrix

Backward Newton-Gregory

$$P_n(x) = f_0 + \binom{s}{1} \cdot \Delta f_{-1} + \binom{s+1}{2} \cdot \Delta^2 f_{-2} + \dots + \binom{s+n-1}{n} \cdot \Delta^n f_{-n} \quad (10)$$

x	$F(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0.2	1.06894					
		0.11242				
0.5	1.18136		0.01183			
		0.12425		0.00123		
0.8	1.30561		0.01306		0.00015	
		0.13731		0.00138		0.000001
1.1	1.44292		0.01444		0.00014	
		0.15175		0.0152		
1.4	1.59467		0.01596			
		0.16771				
1.7	1.76238					

Newton-Gregory forward with $x_0 = 0.5$:

$$P_3(x) = 1.18136 + 0.12425\,s + 0.01306 \binom{s}{2} + 0.00138 \binom{s}{3}$$
$$= 1.18136 + 0.12425\,s + 0.01306\,s\,(s-1)/2 + 0.00138\,s\,(s-1)\,(s-2)/6$$
$$= 0.9996 + 0.3354\,x + 0.052\,x^2 + 0.085\,x^3$$

Newton-Gregory backwards with $x_0 = 1.1$:

$$P_3(x) = 1.44292 + 0.13731\,s + 0.01306 \binom{s+1}{2} + 0.00123 \binom{s+2}{3}$$
$$= 1.44292 + 0.13731\,s + 0.01306\,s\,(s+1)/2 + 0.00123\,s\,(s+1)\,(s+2)/6$$
$$= 0.99996 + 0.33374\,x + 0.05433\,x^2 + 0.007593\,x^3$$

Hermite Polynomial

This applies when we have information not only for the values of $f(x)$ but also on its derivative $f'(x)$

$$P_{2n-1}(x) = \sum_{i=1}^{n} A_i(x) f_i + \sum_{i=1}^{n} B_i(x) f_i' \tag{11}$$

where

$$A_i(x) = [1 - 2(x - x_i)L_i'(x_i)] \cdot [L_i(x)]^2 \tag{12}$$

$$B_i(x) = (x - x_i) \cdot [L_i(x)]^2 \tag{13}$$

and $L_i(x)$ are the Lagrange coefficients.

ERROR: the accuracy is similar to that of Lagrange polynomial of order $2n$!

$$y(x) - p(x) = \frac{y^{(2n+1)}(\xi)}{(2n+1)!} [(x - x_1)(x - x_2) \ldots (x - x_n)]^2 \tag{14}$$

Hermite Polynomial : Example

Fit a Hermite polynomial to the data of the table:

k	x_k	y_k	y'_k
0	0	0	0
1	4	2	0

The Lagrange coefficients are:

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x-4}{0-4} = -\frac{x-4}{4} \qquad L_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x}{4}$$

$$L'_0(x) = \frac{1}{x_0 - x_1} = -\frac{1}{4} \qquad L'_1(x) = \frac{1}{x_1 - x_0} = \frac{1}{4}$$

Thus

$$A_0(x) = \left[1 - 2 \cdot L'_0(x - x_0)\right] \cdot L_0^2 = \left[1 - 2 \cdot \left(-\frac{1}{4}\right)(x - 0)\right] \cdot \left(\frac{x-4}{4}\right)^2$$

$$A_1(x) = \left[1 - 2 \cdot L'_0(x - x_1)\right] \cdot L_1^2 = \left[1 - 2 \cdot \frac{1}{4}(x - 4)\right] \cdot \left(\frac{x}{4}\right)^2 = \left(3 - \frac{x}{2}\right) \cdot \left(\frac{x}{4}\right)^2$$

$$B_0(x) = (x - 0) \cdot \left(\frac{x-4}{4}\right)^2 = x\left(\frac{x-4}{4}\right)^2 \qquad B_1(x) = (x-4) \cdot \left(\frac{x}{4}\right)^2$$

And the Hermite polynomial is:

$$P(x) = (6 - x)\frac{x^2}{16}.$$

Taylor Polynomial

It is an alternative way of approximating functions with polynomials. In the previous two cases we found the polynomial $P(x)$ that gets the same value with a function $f(x)$ at N points or the polynomial that agrees with a function and its derivative at N points. Taylor polynomial has the same value x_0 with the function but agrees also up to the Nth derivative with the given function. That is:

$$P^{(i)}(x_0) = f^{(i)}(x_0) \quad \text{and} \quad i = 0, 1, ..., n$$

and the Taylor polynomial has the well known form from Calculus

$$P(x) = \sum_{i=0}^{N} \frac{f^{(i)}(x)}{i!}(x - x_0)^i \tag{15}$$

ERROR: was also estimated in calculus

$$E_N(x) = (x - x_0)^{N+1} \frac{f^{(N+1)}(\xi)}{(N+1)!} \tag{16}$$

Taylor Polynomial : Example

We will show that for the calculation of $e = 2.718281828459...$ with 13-digit approximation we need 15 terms of the Taylor expansion. All the derivatives at $x = 1$ are:

$$y_0 = y_0^{(1)} = y_0^{(2)} = ... = y_0^{(n)} = 1$$

thus

$$p(x) = \sum_{i=1}^{n} \frac{1}{i!} x^n = 1 + x + \frac{x^2}{2} + ... + \frac{1}{n!} x^n$$

and the error will be

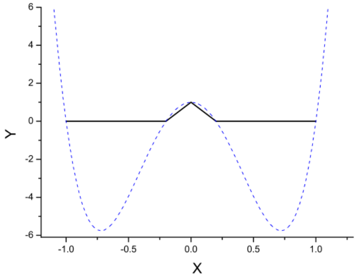
$$|E_n| = x^{n+1} \frac{e^{\xi}}{(n+1)!} = \frac{e^{\xi}}{16!} < \frac{3}{16!} = 1.433 \times 10^{-13}$$

Interpolation with Cubic Splines

In some cases the typical polynomial approximation cannot smoothly fit certain sets of data. Consider the function

$$f(x) = \left[\begin{array}{cc} 0 & -1 \le x \le -0.2 \\ 1 - 5|x| & -0.2 < x < 0.2 \\ 0 & 0.2 \le x \le 1.0 \end{array} \right.$$

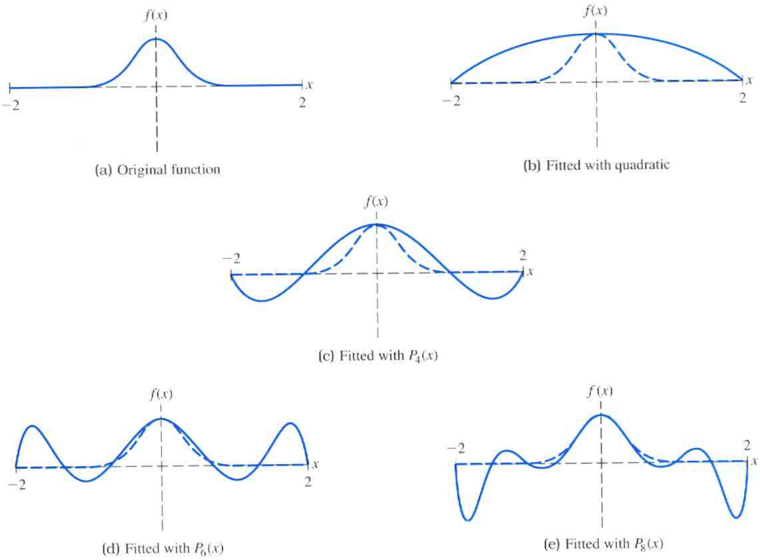
We can easily verify that we cannot fit the above data with any polynomial degree!



$$P(x) = 1 - 26x^2 + 25x^4$$

The answer to the problem is given by the **spline fitting**. That is we pass a set of cubic polynomials (cubic splines) through the points, using a new cubic for each interval. But we require that the slope and the curvature be the same for the pair of cubics that join at each point.

Interpolation with Cubic Splines



Let the cubic for the ith interval, which lies between the points (x_i, y_i) and (x_{i+1}, y_{i+1}) has the form:

$$y(x) = a_i \cdot (x - x_i)^3 + b_i \cdot (x - x_i)^2 + c_i \cdot (x - x_i) + d_i$$

Since it fits at the two endpoints of the interval:

$$y_i = a_i \cdot (x_i - x_i)^3 + b_i \cdot (x_i - x_i)^2 + c_i \cdot (x_i - x_i) + d_i = d_i$$

$$\begin{aligned} y_{i+1} = a_i \cdot (x_{i+1} - x_i)^3 + b_i \cdot (x_{i+1} - x_i)^2 + c_i \cdot (x_{i+1} - x_i) + d_i \\ = a_i h_i^3 + b_i h_i^2 + c_i h_i + d_i \end{aligned}$$

where $h_i = x_{i+1} - x_i$. We need the 1st and 2nd derivatives to relate the slopes and curvatures of the joining polynomials, by differentiation we get

$$\begin{aligned} y'(x) &= 3a_i \cdot (x - x_i)^2 + 2b_i \cdot (x - x_i) + c_i \\ y''(x) &= 6a_i \cdot (x - x_i) + 2b_i \end{aligned}$$

The mathematical procedure is simplified if we write the equations in terms of the 2nd derivatives of the interpolating cubics. Let's name S_i the 2nd derivative at the point (x_i, y_i) then we can easily get:

$$b_i = \frac{S_i}{2}, \quad a_i = \frac{S_{i+1} - S_i}{6h_i} \tag{17}$$

which means

$$y_{i+1} = \frac{S_{i+1} - S_i}{6h_i} h_i^3 + \frac{S_i}{2} h_i^2 + c_i h_i + y_i$$

and finally

$$c_i = \frac{y_{i+1} - y_i}{h_i} - \frac{2h_i S_i + h_i S_{i+1}}{6} \tag{18}$$

Now we invoke the condition that the slopes of the two cubics joining at (x_i, y_i) are the same:

$$\begin{aligned} y_i' &= 3a_i \cdot (x_i - x_i)^2 + 2b_i \cdot (x_i - x_i) + c_i = c_i \\ y_i' &= 3a_{i-1} \cdot (x_i - x_{i-1})^2 + 2b_{i-1} \cdot (x_i - x_{i-1}) + c_{i-1} \\ &= 3a_{i-1} h_{i-1}^2 + 2b_{i-1} h_{i-1} + c_{i-1} \end{aligned}$$

By equating these and substituting a, b, c and d we get:

$$\begin{aligned} y_i' &= \frac{y_{i+1} - y_i}{h_i} - \frac{2h_i S_i + h_i S_{i+1}}{6} \\ &= 3\left(\frac{S_i - S_{i-1}}{6h_{i-1}}\right) h_{i-1}^2 + 2\frac{S_{i-1}}{2} h_{i-1} + \frac{y_i - y_{i-1}}{h_{i-1}} - \frac{2h_{i-1}S_{i-1} + h_{i-1}S_i}{6} \end{aligned}$$

and by simplifying we get:

$$h_{i-1}S_{i-1} + 2\left(h_{i-1} + h_i\right) S_i + h_i S_{i+1} = 6\left(\frac{y_{i+1} - y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}}\right) \quad (19)$$

If we have $n+1$ points the above relation can be applied to the $n-1$ internal points. Thus we create a system of $n-1$ equations for the $n+1$ unknown S_i. This system can be solved if we specify the values of S_0 and S_n.

The system of $n-1$ equations with $n+1$ unknown will be written as:

$$\begin{pmatrix} & & & \cdots\cdots & & & \\ h_0 & 2(h_0+h_1) & h_1 & & & & \\ & h_1 & 2(h_1+h_2) & h_2 & & & \\ & & h_2 & 2(h_2+h_3) & h_3 & & \\ & & & \cdots\cdots & \cdots\cdots & & \\ & & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ \vdots \\ S_{n-2} \\ S_{n-1} \\ S_n \end{pmatrix}$$

$$= 6 \begin{pmatrix} \cdots \\ \frac{y_2-y_1}{h_1} - \frac{y_1-y_0}{h_0} \\ \frac{y_3-y_2}{h_2} - \frac{y_2-y_1}{h_1} \\ \cdots \\ \frac{y_n-y_{n-1}}{h_{n-1}} - \frac{y_{n-1}-y_{n-2}}{h_{n-2}} \\ \cdots \end{pmatrix} \equiv \vec{Y}$$

From the solution of this linear systems we get the coefficients a_i, b_i, c_i and d_i via the relations:

$$a_i = \frac{S_{i+1}-S_i}{6h_i}, \quad b_i = \frac{S_i}{2} \tag{20}$$

$$c_i = \frac{y_{i+1}-y_i}{h_i} - \frac{2h_iS_i + h_iS_{i+1}}{6} \tag{21}$$

$$d_i = y_i \tag{22}$$

• **Choice I** Take, $S_0 = 0$ and $S_n = 0$ this will lead to the solution of the following $(n-1) \times (n-1)$ linear system:

$$\begin{pmatrix} 2(h_0+h_1) & h_1 & & & \\ h_1 & 2(h_1+h_2) & h_2 & & \\ & h_2 & 2(h_2+h_3) & h_3 & \\ & & .... & & \\ & & h_{n-2} & & 2(h_{n-2}+h_{n-1}) \end{pmatrix} \cdot \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ \\ S_{n-1} \end{pmatrix} = \vec{Y}$$

• **Choice II** Take, $S_0 = S_1$ and $S_n = S_{n-1}$ this will lead to the solution of the following $(n-1) \times (n-1)$ linear system:

$$\begin{pmatrix} 3h_0+2h_1 & h_1 & & & \\ h_1 & 2(h_1+h_2) & h_2 & & \\ & h_2 & 2(h_2+h_3) & h_3 & \\ & & \cdots & \cdots & \\ & & & h_{n-2} & 2h_{n-2}+3h_{n-1} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ \\ S_{n-1} \end{pmatrix} = \vec{Y}$$

- **Choice III** Use linear extrapolation

$$\frac{S_1 - S_0}{h_0} = \frac{S_2 - S_1}{h_1} \Rightarrow S_0 = \frac{(h_0 + h_1)S_1 - h_0 S_2}{h_1}$$

$$\frac{S_n - S_{n-1}}{h_{n-1}} = \frac{S_{n-1} - S_{n-2}}{h_{n-2}} \Rightarrow s_n = \frac{(h_{n-2} + h_{n-1})S_{n-1} - h_{n-1}S_{n-2}}{h_{n-2}}$$

this will lead to the solution of the following $(n-1) \times (n-1)$ linear system:

$$\begin{pmatrix} \frac{(h_0+h_1)(h_0+2h_1)}{h_1} & \frac{h_1^2-h_0^2}{h_1} & & \\ h_1 & 2(h_1+h_2) & h_2 & \\ & h_2 & 2(h_2+h_3) & h_3 \\ & & \cdots & \cdots \\ & & \frac{h_{n-2}^2-h_{n-1}^2}{h_{n-2}} & \frac{(h_{n-1}+h_{n-2})(h_{n-1}+2h_{n-2})}{h_{n-2}} \end{pmatrix} \cdot \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ \\ S_{n-1} \end{pmatrix} = \vec{Y}$$

• **Choice IV** Force the slopes at the end points to assume certain values. If $f'(x_0) = A$ and $f'(x_n) = B$ then

$$\begin{aligned} 2h_0 S_0 + h_1 S_1 &= 6\left(\frac{y_1 - y_0}{h_0} - A\right) \\ h_{n-1} S_{n-1} + 2h_n S_n &= 6\left(B - \frac{y_n - y_{n-1}}{h_{n-1}}\right) \end{aligned}$$

$$\begin{pmatrix} 2h_0 & h_1 & & & \\ h_0 & 2(h_0+h_1) & h_1 & & \\ & h_1 & 2(h_1+h_2) & h_2 & \\ & & \cdots\cdot & & \\ & & & h_{n-2} & 2h_{n-1} \end{pmatrix} \cdot \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ \\ S_{n-1} \end{pmatrix} = \vec{Y}$$

Fit a cubic spline in the data ($y = x^3 - 8$):

x	0	1	2	3	4
y	-8	-7	0	19	56

Depending on the condition at the end we get the following solutions:

- **Condition I** : $S_0 = 0,\ S_4 = 0$

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix} \cdot \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} 36 \\ 72 \\ 108 \end{pmatrix} \Rightarrow \begin{aligned} S_1 &= 6.4285 \\ S_2 &= 10.2857 \\ S_3 &= 24.4285 \end{aligned}$$

- **Condition II** : $S_0 = S_1,\ S_4 = S_3$

$$\begin{pmatrix} 5 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 5 \end{pmatrix} \cdot \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} 36 \\ 72 \\ 108 \end{pmatrix} \Rightarrow \begin{aligned} &S_1 = S_0 = 4.8 \\ &S_2 = 12 \\ &S_3 = 19.2 = S_4 \end{aligned}$$

- **Condition III** :

$$\begin{pmatrix} 6 & 0 & 0 \\ 1 & 4 & 1 \\ 0 & 0 & 6 \end{pmatrix} \cdot \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} 36 \\ 72 \\ 108 \end{pmatrix} \Rightarrow \quad \begin{array}{l} S_0 = 0 \;\; S_1 = 6 \\ S_2 = 12 \;\; S_3 = 18 \\ S_4 = 24 \end{array}$$

- **Condition IV** :

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 36 \\ 72 \\ 108 \\ 66 \end{pmatrix} \quad \begin{array}{l} S_0 = 0 \\ S_1 = 6 \\ S_2 = 12 \\ S_3 = 18 \\ S_4 = 24 \end{array}$$

1. The following data are from astronomical observations and represent variations of the apparent magnitude of a type of variable stars called Cepheids

Time	0.0	0.2	0.3	0.4	0.5	0.6	0.7	0.8	1.0
Apparent magnitude	0.302	0.185	0.106	0.093	0.24	0.579	0.561	0.468	0.302

   Use splines to create a new table for the apparent magnitude for intervals of time of 0.5.

2. From the following table find the acceleration of gravity at Tübingen (48^o 31') and the distance between two points with angular separation of 1' of a degree.

Latitude	Length of 1' of arc on the parallel	local acceleration of gravity g
0^0	1855.4 m	9.7805 m/sec^2
15^0	1792.0 m	9.7839 m/sec^2
30^0	1608.2 m	9.7934 m/sec^2
45^0	1314.2 m	9.8063 m/sec^2
60^0	930.0 m	9.8192 m/sec^2
75^0	481.7 m	9.8287 m/sec^2
90^0	0.0 m	9.8322 m/sec^2

Rational function approximations

Here we introduce the notion of **rational approximations for functions**. We will constrain our discussion to the so called **Padé approximation**. A rational approximation of $f(x)$ on $[a, b]$, is the quotient of two polynomials $P_n(x)$ and $Q_m(x)$ with degrees n and m[1]

$$f(x) = R_N(x) \equiv \frac{P_n(x)}{Q_m(x)} = \frac{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n}{1 + b_1 x + b_2 x^2 + \ldots + b_m x^m}, \quad N = n + m$$

i.e. there are $N + 1 = n + m + 1$ constants to be determined. The method of Padé requires that $f(x)$ and its derivatives are continuous at $x = 0$. This choice makes the manipulation simpler and a change of variable can be used to shift, if needed, the calculations over to an interval that contains zero.
We begin with the Maclaurin series for $f(x)$ (up to the term x^N), this can be written as :

$$f(x) \approx c_0 + c_1 x + \ldots + c_N x^N$$

where $c_i = f^{(i)}(0)/(i!)$.

[1]Sometimes we write $R_N(x) \equiv R_{n,m}(x)$.

Padé approximation

Then we create the difference

$$
\begin{aligned}
f(x) - R_N(x) &\approx \left(c_0 + c_1 x + \ldots + c_N x^N\right) - \frac{a_0 + a_1 x + \ldots + a_n x^n}{1 + b_1 x + \ldots + b_m x^m} \\
&= \frac{\left(c_0 + c_1 x + \ldots + c_N x^N\right)\left(1 + b_1 x + \ldots + b_m x^m\right) - \left(a_0 + a_1 x + \ldots + a_n x^n\right)}{1 + b_1 x + \ldots + b_m x^m}
\end{aligned}
$$

If $f(0) = R_N(0)$ then $c_0 - a_0 = 0$.
In the same way , in order for the first N derivatives of $f(x)$ and $R_N(x)$ to be equal at $x = 0$ the coefficients of the powers of x up to x^N in the numerator must be zero also.

This gives additionally N equations for the a's and b's

$$
\begin{aligned}
b_1 c_0 + c_1 - a_1 &= 0 \\
b_2 c_0 + b_1 c_1 + c_2 - a_2 &= 0 \\
b_3 c_0 + b_2 c_1 + b_1 c_2 + c_3 - a_3 &= 0 \\
&\vdots \\
b_m c_{n-m} + b_{m-1} c_{n-m+1} + \ldots + c_n - a_n &= 0 \\
b_m c_{n-m+1} + b_{m-1} c_{n-m+2} + \ldots + c_{n+1} &= 0 \\
b_m c_{n-m+2} + b_{m-1} c_{n-m+3} + \ldots + c_{n+2} &= 0 \\
&\vdots \\
b_m c_{N-m} + b_{m-1} c_{N-m+1} + \ldots + c_N &= 0
\end{aligned}
\tag{23}
$$

Notice that in each of the above equations, the sum of subscripts on the factors of each product is the same, and is equal to the exponent of the x-term in the numerator.

Padé approximation : Example

For the $R_9(x)$ or $R_{5,4}(x)$ Padé approximation for the function $\tan^{-1}(x)$ we calculate the Maclaurin series of $\tan^{-1}(x)$:

$$\tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9$$

and then

$$f(x) - R_9(x) = \frac{\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9\right)\left(1 + b_1x + b_2x^2 + b_3x^3 + b_4x^4\right) - \left(a_0 + a_1x + a_2x^2 + \ldots + a_5x^5\right)}{1 + b_1x + b_2x^2 + b_3x^3 + b_4x^4}$$

and the coefficients will be found by the following system of equations:

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = b_1, \quad a_3 = -\frac{1}{3} + b_2, \quad a_4 = -\frac{1}{3}b_1 + b_3, \quad a_5 = \frac{1}{5} - \frac{1}{3}b_2 + b_4$$
$$\frac{1}{5}b_1 - \frac{1}{2}b_3 = 0, \quad -\frac{1}{7} + \frac{1}{5}b_2 - \frac{1}{3}b_4 = 0, \quad -\frac{1}{7}b_1 + \frac{1}{5}b_3 = 0, \quad \frac{1}{9} - \frac{1}{7}b_2 + \frac{1}{5}b_4 = 0 \qquad (24)$$

from which we get

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 0, \quad a_3 = \frac{7}{9}, \quad a_4 = 0, \quad a_5 = \frac{64}{945}, \quad b_1 = 0, \quad b_2 = \frac{10}{9}, \quad b_3 = 0, \quad b_4 = \frac{5}{21}.$$

$$\tan^{-1}x \approx R_9(x) = \frac{x + \frac{7}{9}x^3 + \frac{64}{945}x^5}{1 + \frac{10}{9}x^2 + \frac{5}{21}x^4}$$

For $x = 1$, exact 0.7854, $R_9(1) = 0.78558$ while from the Maclaurin series we get 0.8349!

If instead of the analytic form of a function $f(x)$ we have a set of k points $(x_i, f(x_i))$ in order to find a rational function $R_N(x)$ such that for every x_i we will get $f(x_i) = R_N(x_i)$ i.e.

$$R_N(x_i) = \frac{a_0 + a_1 x_i + a_2 x_i^2 + \ldots + a_n x_i^n}{1 + b_1 x_i + b_2 x_i^2 + \ldots + b_m x_i^m} = f(x_i)$$

we will follow the approach used in constructing the approximate polynomial. In other words the problem will be solved by finding the solution of the following system of $k \geq m + n + 1$ equations:

$$\begin{array}{l} a_0 + a_1 x_1 + \ldots + a_n x_1^n - (f_1 x_1)\ b_1 - \ldots - (f_1 x_1^m)\ b_m = f_1 \\ :: \\ a_0 + a_1 x_i + \ldots + a_n x_i^n - (f_i x_i)\ b_1 - \ldots - (f_i x_i^m)\ b_m = f_i \\ :: \\ a_0 + a_1 x_k + \ldots + a_n x_k^n - (f_k x_k)\ b_1 - \ldots - (f_k x_k^m)\ b_m = f_k \end{array}$$

i.e. we get k equations for the k unknowns $a_{0,} a_1, \ldots, a_n$ and $b_1, b_2, \ldots, b_m$.

Rational approximation for sets of data : Example

We will find the rational function approximations for the following set of data (-1,1), (0,2) and (1,-1).

It is obvious that the sum of degrees of the polynomials in the nominator and denominator must be $(n+m+1 \leq 3)$. Thus we can write:

$$R_{1,1}(x) = \frac{a_0 + a_1 x}{1 + b_1 x}$$

which leads to the following system

$$\left.\begin{array}{l} a_0 + (-1)\ a_1 - (-1)\ b_1 = 1 \\ a_0 + 0 \cdot a_1 - 0 \cdot\ b_1 = 2 \\ a_0 + 1 \cdot\ a_1 - (-1)\ b_1 = -1 \end{array}\right\} \Rightarrow \qquad \begin{array}{l} a_0 = 2 \\ a_1 = -1 \\ b_1 = -2 \end{array}$$

and the rational fuction will be:

$$R_{1,1}(x) = \frac{2 - x}{1 - 2x}.$$

Alternatively, one may derive the following rational function:

$$R_{0,1}(x) = \frac{a_0}{1 + b_1 x + b_2 x} \Rightarrow R(x) = \frac{2}{1 - 2x - x^2}$$

Rational approximation: Problems

1. Find the Padé approximation $R_{3,3}(x)$ for the function $y = e^x$. Compare with the Maclaurin series for $x = 1$.
2. Find the Padé approximation $R_{3,5}(x)$ for the functions $y = \cos(x)$ and $y = \sin(x)$. Compare with the Maclaurin series for $x = 1$.
3. Find the Padé approximation $R_{4,6}(x)$. for the function $y = 1/x \sin(x)$. Compare with the Maclaurin series for $x = 1$.
4. Find the rational approximation for the following set of points:

0	1	2	4
0.83	1.06	1.25	4.15